SU(2)-symmetries and exact sequences of C*-algebras through subproduct systems

Francesca Arici

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Universiteit Leiden
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The circle:

$$S^1 := \{z \in \mathbb{C} \mid \bar{z}z = 1\}.$$

The C*-algebra $C(S^1)$ is the closure of the *Laurent polynomials*

$$\frac{\mathbb{C}[\zeta, \bar{\zeta}]}{\langle \bar{\zeta}\zeta = 1 \rangle}.$$

We represent $C(S^1)$ via multiplication operators on the Hilbert space

$$H = L^2(S^1) \simeq \ell^2(\mathbb{Z}).$$

Under this isomorphism, multiplication by $e^{2\pi \mathrm{i}\theta}$ is mapped to the bilateral shift

$$U(e_n) = (e_{n+1}), \quad U^*(e_n) = e_{n-1}.$$

$C(S^1)$ is the smallest C*-subalgebra of $B(\ell^2(\mathbb{Z}))$ that contains the *unitary* U.

Now instead consider the Hilbert space $\ell^2(\mathbb{N})$ and the shift operator

$$T(e_n) = (e_{n+1}) \qquad T^*(e_n) = \begin{cases} e_{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases}.$$

The *Toeplitz algebra* $\mathcal{T}$ is the smallest C*-subalgebra of $B(\ell^2(\mathbb{N}))$ that contains T.

It is not commutative since $T^*T = \mathrm{Id}$ and $TT^* = 1 - P_{\ker(T^*)}$.
Elements of $\mathcal{T}$ commute up to compact operators:

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\pi} C(S^1) \longrightarrow 0.$$

Consequences: Noether–Goberg–Krein index theorem, Cuntz's proof of Bott periodicity, PV-exact sequence for crossed products by $\mathbb{Z}$, and more.
The algebra $C(S^1)$ is the "boundary" of a noncommutative disk.

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\pi} C(S^1) \longrightarrow 0.$$

Cuntz Pimsner algebras of (injective) C*-correspondences.

$$0 \longrightarrow J(X) \xrightarrow{j} \mathcal{T}_X \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0.$$

Arveson's Toeplitz extensions for odd-dimensional spheres.

$$0 \longrightarrow \mathcal{K}(H^2_d) \xrightarrow{j} \mathcal{T}_d \xrightarrow{\pi} C(S^{2d-1}) \longrightarrow 0.$$

All these are examples of the *defining extensions* for Cuntz–Pimsner algebras of **subproduct systems** (Shalit and Solel 2009, Viselter 2012).

Cuntz–Pimsner algebras: universal C*-algebras associated to an (injective) C*-correspondence (X_A, ϕ).

Special case: Cuntz–Pimsner algebras of self-Morita equivalence bimodules (noncommutative line bundles), i.e. correspondences with $\phi : A \xrightarrow{\simeq} \mathbb{K}(X_A)$.

- Resulting Cuntz–Pimsner algebra: noncommutative *associated* circle bundle over A (cf. A–Kaad–Landi 2016).
- The corresponding six-term exact sequence in K-theory

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-[X]} & K_0(A) & \xrightarrow{j_*} & K_0(\mathcal{O}_X) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ K_1(\mathcal{O}_X) & \xleftarrow[j_*]{} & K_1(A) & \xleftarrow[1-[X]]{} & K_1(A) \end{array},$$

  is a noncommutative analogue of the Gysin sequence for circle bundles.
- Multiplication by the Euler class replaced by Kasparov product with $(1-[X]) \in KK(A,A)$.

Let $d \in \mathbb{N}_0$, and $z_0, \ldots, z_{d-1}$ commuting variables, and consider the space of polynomials $\mathbb{C}[z_0, \ldots, z_{d-1}]$.
For $z = (z_0, ..., z_{d-1})$ and every multi-index $\alpha = (\alpha_0, \ldots, \alpha_{d-1}) \in \mathbb{N}_0^d$ we write

$$z^\alpha = z_0^{\alpha_0} \cdots z_{d-1}^{\alpha_{d-1}}.$$

The Drury–Arveson space H_d^2 is a completion of the polynomials $\mathbb{C}[z_0, \ldots, z_{d-1}]$, w.r.t. the inner product

$$\langle z^\alpha, z^\beta \rangle = \delta_{\alpha,\beta} \frac{\alpha!}{|\alpha!|}$$

It can be identified with the space of holomorphic functions $f : \mathbb{B}^d \subseteq \mathbb{C}^d \to \mathbb{C}$ which have a power series $f(z) = \sum_\alpha c_\alpha z^\alpha$ satisfying

$$\|f\|_d^2 := \sum_\alpha |c_\alpha|^2 \frac{\alpha!}{|\alpha!|} < \infty.$$

Clearly, $H_d^2 \simeq \mathbb{F}_{\mathrm{sym}}(\mathbb{C}^d) := \bigoplus_{n \geq 0} \mathrm{Sym}^n(\mathbb{C}^d)$, the d-symmetric Fock space.

The symmetric Fock space is a subset of the full Fock space: $\mathbb{F}_{\rm sym}(\mathbb{C}^d) \subseteq \mathbb{F}(\mathbb{C}^d) := \bigoplus_{n\geq 0}(\mathbb{C}^d)^{\otimes n}$.
On H^2_d, we consider the d-shift, a d-tuple of multiplication operators given by

$$Mz = (M_{z_0}, \dots, M_{z_{d-1}}).$$

Through $H^2_d \simeq \mathbb{F}_{\rm sym}(\mathbb{C}^d)$, the shift operator is identified with a compression of the shift on the full Fock space:
Let $\{e_i\}_{i=0}^{d-1}$ be an orthonormal basis for $\mathbb{C}^d$, and let $P : \mathbb{F}(\mathbb{C}^d) \to \mathbb{F}_{\rm sym}(\mathbb{C}^d)$ the orthogonal projection.
For $x \in \mathrm{Sym}^n(\mathbb{C}^d)$, and for all $i = 0, \cdots, d-1$, have the compressed shift

$$T_i(x) = P(e_i \otimes x).$$

The d-tuple $T = (T_0, \dots, T_{d-1})$ is called the *commutative* d-shift.
If $V : H^2_d \to \mathbb{F}_{\rm sym}(\mathbb{C}^d)$ is the unitary implementing the isomorphism, we have $VM_zV^* = T$.

The d-shift satisfies the following properties:

- T is commuting: $T_i T_j = T_j T_i$.
- $\sum_{i=0}^{d-1} T_i T_i^* = 1 - P_{\mathbb{C}}$
- T is essentially normal:
$$T_i^* T_j - T_j T_i^* = (1+N)^{-1}(\delta_{ij}1 - T_j T_i^*),$$
where N is the number operator: $N\xi = n\xi$ for $\xi \in \mathrm{Sym}^n(\mathbb{C}^d)$.

Theorem (Arveson 1998)

Let $\mathbb{T}_d = C^*(1, T)$ be the C^*-algebra generated by the d-shift. We have an exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}(H_d^2) \longrightarrow \mathbb{T}_d \longrightarrow C(S^{2d-1}) \longrightarrow 0\ , \tag{1}$$

where $C(S^{2d-1})$ is the commutative C^*-algebra of continuous functions on the $(2d-1)$-sphere $S^{2d-1} = \partial\mathbb{B}^d \subseteq \mathbb{C}^d$.

motivation ◂

Suppose that $\{E_m\}_{m\in\mathbb{N}_0}$ is a sequence of Hilbert spaces with $\iota_{k,m} : E_{k+m} \to E_k \widehat{\otimes} E_m$ is a sequence of bounded isometries for every $k, m \in \mathbb{N}_0$.

We say that (E, ι) is a *standard subproduct system* over $\mathbb{C}$ when the following holds for all $k, l, m \in \mathbb{N}_0$:

1. $E_0 = \mathbb{C}$;
2. The maps $\iota_{0,m} : E_m \to E_0 \widehat{\otimes} E_m$ and $\iota_{m,0} : E_m \to E_m \widehat{\otimes} E_0$ are the canonical identifications; and
3. We have

$$(1_k \otimes \iota_{l,m}) \circ \iota_{k,l+m} = (\iota_{k,l} \otimes 1_m) \circ \iota_{k+l,m}$$

on $E_{k+l+m} \to E_k \widehat{\otimes} E_l \widehat{\otimes} E_m$, where 1_k and 1_m denote the identity operators on E_k and E_m, resp.

We refer to the isometries $\iota_{k,m} : E_{k+m} \to E_k \widehat{\otimes} E_m$, $k, m \in \mathbb{N}_0$ as the structure maps of our subproduct system.

Let $X_n := \{x_0, \dots, x_n\}$ be a finite set of variables. We shall denote the free monoid generated by X_n by $\langle X \rangle$, with unit the empty word, denoted by 1. $\langle X \rangle$ is naturally graded by length :

$$\langle X \rangle = \bigsqcup_{m \geq 0} X^m.$$

Let $\mathbb{C}\langle X_n \rangle := \mathbb{C}\langle x_0, \dots, x_n \rangle$ denote the complex free associative algebra with unit generated by X_n.
An element of $\mathbb{C}\langle X \rangle$ is called a noncommutative polynomial. A polynomial $f \in \mathbb{C}\langle X \rangle$ is *homogeneous of degree* m if $f \in \mathbb{C}X^m$.
Let $T = (T_i)_{i=0}^n$ be an $(n+1)$-tuple of operators acting on a Hilbert space. If $\alpha = (\alpha_1, \dots, \alpha_m) \in X_m$ is a length m word, then

$$T^\alpha := T_{\alpha_1} \dots T_{\alpha_k},$$

with the convention that $T^1 = 1_H$.
If $p(x) = \sum c_\alpha x^\alpha \in \mathbb{C}\langle X \rangle$ is a complex polynomial, by $p(T)$ we mean the linear combination $p(T) := \sum c_\alpha T^\alpha$.

Theorem (Shalit–Solel 2009)

Let E be an $(n+1)$-dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=0}^n$. Then there is a bijective inclusion-reversing correspondence between proper homogeneous ideals $J \lhd \mathbb{C}\langle x_0, \ldots, x_n\rangle$ and standard subproduct systems $E = \{E_m\}_{m\in\mathbb{N}_0}$ with $E_1 \subseteq E$.

Notation: for a polynomial $p = \sum c_\alpha x^\alpha$, we write $p(e) = \sum c_\alpha e_{\alpha_0} \otimes \ldots \otimes e_{\alpha_n}$.

$$J \lhd \mathbb{C}\langle X\rangle \longrightarrow E_n^J := E^{\otimes n} \ominus \{p(e) | p \in J^{(n)}\}$$

$$E = \{E_m\}_{m\in\mathbb{N}_0} \longrightarrow J^E = \mathrm{span}\{p \in \mathbb{C}\langle X\rangle \mid \exists n > 0 : p(e) \in E^{\otimes n} \ominus E_n\}$$

Framework for studying tuples of operators satisfying relations given by homogeneous polynomials (cf. Popescu's work on noncommutative domains).

Let E be a standard subproduct system of Hilbert spaces over $\mathbb{N}_0$.
The direct sum Hilbert space $F_E := \bigoplus_{m\geq 0} E_m$ is called the *Fock space*.
For each $\xi \in E_k$, we define the Toeplitz operator on F_E:

$$T_\xi : F_E \to F_E \quad T_\xi(\zeta) := \iota^*_{k,m}(\xi \otimes \zeta), \quad \zeta \in E_m \subseteq F_E.$$

Definition

The *Toeplitz algebra* of the subproduct system E, denoted $\mathbb{T}_E$ is the smallest unital C^*-subalgebra of $\mathbb{L}(F)$ that contains all the creation operators, i.e.

$$T_\xi \in \mathbb{T}_E \quad \text{for all } \xi \in E_k \ , \ \ k \in \mathbb{N}_0.$$

Note that the definition makes sense in the case of C*-correspondences.

If X is a subproduct system of finite dimensional Hilbert spaces, then the algebra of compact operators $\mathcal{K}(F_E)$ is naturally included in $\mathbb{T}_E$.

Definition

Let E be a subproduct system of finite dimensional Hilbert spaces. The Cuntz–Pimsner algebra of the subproduct system E is the quotient

$$0 \longrightarrow \mathbb{K}(F_E) \longrightarrow \mathbb{T}_E \xrightarrow{q} \mathbb{O}_E \longrightarrow 0. \tag{2}$$

Question: can we say something about the exact sequences this extension induces?

Let $\tau : SU(2) \to U(H)$ be a strongly continuous unitary representation, $\dim(H) < \infty$. Define

$$\det(\tau, H) = \{\xi \in H \otimes H \mid \big(\tau(g) \otimes \tau(g)\big)\xi = \xi \quad \forall g \in SU(2)\}.$$

For a given $n \geq 0$, consider the irreducible representation $\rho_n : SU(2) \to U(L_n)$.
Here $L_n = (\mathbb{C}^2)^{\otimes_S n}$, with orthonormal basis $\{e_k\}_{k=0}^n$.

Suppose that $\tau : SU(2) \to U(H)$ is irreducible and let $V : L_n \to H$ be a unitary operator intertwining τ with ρ_n.
Then the determinant $\det(\tau, H) \subseteq H \otimes H$ is a one-dimensional vector space spanned by the vector

$$(V \otimes V)\Big((n+1)^{-1/2} \sum_{k=0}^{n} (-1)^{n-k} e_k \otimes e_{n-k}\Big).$$

For a given $n \geq 0$, consider the *irreducible* representation $\rho_n : SU(2) \to U(L_n)$. Where $L_n = (\mathbb{C}^2)^{\otimes_S n}$. Let $D := \det(\rho_n, L_n) \subseteq L_n \otimes L_n$. We inductively construct a subproduct system X where

- $E_0 = \mathbb{C}$;
- $E_1 = L_n$;
- $E_m := K_m^{\perp} \subseteq (L_n)^{\otimes m}$, where

$$K_m = \sum_{i=0}^{m-2} L_n^{\otimes i} \otimes D \otimes L_n^{\otimes (m-i-2)}$$

The structure maps $\iota_{k,m} : E_{k+m} \to E_k \otimes E_m$, $k, m \in \mathbb{N}_0$, induced by the canonical identification $L^{\otimes (k+m)} \cong L^{\otimes k} \otimes L^{\otimes m}$ are are $SU(2)$-equivariant.

The pair $(E = \{E_m\}_{m=0}^{\infty}, \iota)$ forms an $SU(2)$-equivariant subprouduct system of finite dimensional Hilbert spaces.

Let us denote by $\{e_0, e_1\}$ the standard basis for $\mathbb{C}^2$, and let $\rho : SU(2) \to U(\mathbb{C}^2)$ the fundamental representation. We have that

$$\det(\rho, \mathbb{C}^2) = \mathbb{C} \cdot (e_0 \otimes e_1 - e_1 \otimes e_0) \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$$

Remark in particular that $\det(\rho, \mathbb{C}^2)$ agrees with the usual determinant of $\mathbb{C}^2$ namely $\mathbb{C}^2 \wedge \mathbb{C}^2 \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$.
Let now $m \in \mathbb{N}$ and consider the m-fold symmetric tensor product of a finite dimensional Hilbert space H.
It follows from the Clebsch–Gordan theory for the representations of $SU(2)$.

$$(\mathbb{C}^2)^{\otimes_S m} = E_m(\rho, \mathbb{C}^2).$$

For each $m \in \mathbb{N}$, let $p_m : (\mathbb{C}^2)^{\otimes m} \to (\mathbb{C}^2)^{\otimes m}$ denotes the orthogonal projection onto $\mathrm{Sym}^m(\mathbb{C}^2)$. The vectors

$$e_0^k e_1^{m-k} := p_m(e_0^{\otimes k} \otimes e_1^{\otimes (m-k)}), \quad k = 0, \ldots, m,$$

form an orthogonal vector space basis for $E_m(\rho, \mathbb{C}^2)$ (cf. Arveson's d-shift).

Let τ be an irreducible $SU(2)$ representation, and let $\{E_m\}_{m\geq 0}$ be the associated $SU(2)$-subproduct system.

Theorem

For each $k, m \in \mathbb{N}_0$ there exists an explicit $SU(2)$-equivariant unitary isomorphism

$$E_k \otimes E_m \cong E_{k+m} \oplus \ldots \oplus E_{k+m-2} \oplus \ldots \oplus E_{|k-m|}.$$

We consider the sequence of strictly positive integers $\{d_m\}_{m=0}^{\infty}$ defined recursively by the formula:

$$d_0 := 1 \ , \ \ d_1 := n+1 \ , \ \ d_m := d_1 \cdot d_{m-1} - d_{m-2} \ , \ \ m \geq 2. \tag{3}$$

We furthermore put $d_{-1} := 0$

We have that $\dim(E_m) = d_m$ for all $m \in \mathbb{N}_0$.

We construct the Fock space $F_E := \bigoplus_{m \geq 0} E_m(\rho_n, L_n)$.
We let $\{e_j\}_{j=0}^n$ denote the orthonormal basis for L_n and consider the associated Toeplitz operators:

$$T_i := T_{e_i} : F_E \to F_E \quad T_i(\zeta) := \iota_{1,m}^*(e_i \otimes \zeta), \quad \zeta \in E_m(\rho_n, L_n).$$

where $\iota_{1,m} : E_{m+1} \to E_1 \otimes E_m$, for $m \in \mathbb{N}_0$.

The assignment

$$g(T_i) := T_{g \cdot e_i}, \qquad \forall g \in SU(2)$$

defines a strongly continuous action of $SU(2)$ on the Toeplitz algebra $\mathbb{T}_E$.

Denote by $F_{\mathrm{alg}} \subseteq F$ the algebraic Fock-space.

We define the *dimension operator* $D : \mathrm{Dom}(D) \to F$ as the closure of the unbounded operator $\mathcal{D} : F_{\mathrm{alg}} \to F$, given by $\mathcal{D}(\xi) = d_m \cdot \xi$ for $\xi \in E_m$.

The dimension operator is positive and invertible and that the inverse $D^{-1} : F \to F$ is an $SU(2)$-equivariant compact operator, so $D^{-1} \in \mathbb{T}_E$.
For the fundamental representation, the operator D equals $N + 1$, where N is the number operator.

Further, we have an $SU(2)$-equivariant bounded positive invertible operator

$$\Phi : F \to F \quad \Phi\xi = \frac{d_m}{d_{m+1}}\xi \text{ for all } \xi \in E_m. \tag{4}$$

Theorem

Let $n \in \mathbb{N}_0$, and consider the irreducible representation $\rho_n : SU(2) \to U(L_n)$. Then the Toeplitz operators satisfy the following commutation relations:

$$\sum_{i=0}^{n} (-1)^i T_i T_{n-i} = 0, \tag{5}$$

$$T_i^* T_j = \delta_{ij} 1 + (-1)^{i+j+1}((n+1) - \Phi^{-1}) T_{n-i} T_{n-j}^*, \qquad i \le j \tag{6}$$

$$\sum_{i=0}^{n} T_i^* T_i = \Phi^{-1}. \tag{7}$$

For $n = 1$ we get back the commutation relations for the Arveson–Toeplitz algebra $\mathbb{T}_2$.

On the Hilbert space direct sum $F \oplus F$, we consider two $*$-homomorphisms $\psi_\pm : \mathbb{T} \to \mathbb{L}(F \oplus F)$:

$$\psi_+(x) := x \oplus x, \qquad \text{for all } x \in \mathbb{T}_E.$$

The construction of $\psi_- : \mathbb{T}_E \to \mathbb{L}(F \oplus F)$ uses representation theoretic considerations: we use the $SU(2)$-equivariant unitary isomorphism $E_{m+1} \oplus E_{m-1} \simeq E_m \otimes E_1$ to construct an $SU(2)$-equivariant isometry

$$W_R : F \otimes E_1 \to F \oplus F \tag{8}$$

with image $F_+ \oplus F \subseteq F \oplus F$.
We may thus define the $*$-homomorphism

$$\psi_- : \mathbb{T} \to \mathbb{L}(F \oplus F) \quad \psi_-(x) := W_R(x \otimes 1)W_R^*.$$

Proposition

The pair of $*$-homomorphisms (ψ_+, ψ_-) defines a class $[\psi_+, \psi_-] \in KK_0^{SU(2)}(\mathbb{T}, \mathbb{C})$.

We have a class $[\psi_+, \psi_-] \in KK_0^{SU(2)}(\mathbb{T}, \mathbb{C})$.
The canonical inclusion $i : \mathbb{C} \to \mathbb{T}$ yields a class $[i] \in KK_0^{SU(2)}(\mathbb{C}, \mathbb{T})$.

Theorem (A–Kaad 2020)

Let $\mathbb{T}_E$ be the Toeplitz algebra of the $SU(2)$-product system of an irreducible representation. Then $\mathbb{T}_E$ and $\mathbb{C}$ are KK-equivalent (in an $SU(2)$-equivariant way).

$$\boxed{[i]\,\widehat{\otimes}_{\mathbb{T}}[\psi_+, \psi_-] = \mathbf{1}_{\mathbb{C}} \in KK_0^{SU(2)}(\mathbb{C}, \mathbb{C})}$$

"Easy" direction.

$$\boxed{[\psi_+, \psi_-]\widehat{\otimes}_{\mathbb{C}}[i] = \mathbf{1}_{\mathbb{T}} \in KK_0^{SU(2)}(\mathbb{T}, \mathbb{T})}$$

Requires a homotopy argument. Note that we do not have universal properties for $\mathbb{T}_E$ (yet).

The extension

$$0 \longrightarrow \mathbb{K}(F_E) \xrightarrow{j} \mathbb{T}_E \xrightarrow{q} \mathbb{O}_E \longrightarrow 0. \tag{9}$$

Induces six term exact sequences

$$\begin{array}{ccccc}
K_0(\mathbb{K}(F_E)) & \xrightarrow{j_*} & K_0(\mathbb{T}_E) & \xrightarrow{q_*} & K_0(\mathbb{O}_E) \\
\uparrow & & & & \downarrow \\
K_1(\mathbb{O}_E) & \xleftarrow[q_*]{} & K_1(\mathbb{T}_E) & \xleftarrow[j_*]{} & K_0(\mathbb{K}(F_E))
\end{array}$$

Goal: simplify using the KK-equivalence between $\mathbb{T}_E$ and $\mathbb{C}$, and the Morita equivalence between $\mathbb{C}$ and $\mathbb{K}(F_E)$.

We have the identity

$$[j]\widehat{\otimes}_{\mathbb{T}}[\psi_+, \psi_-] = [F]\widehat{\otimes}_{\mathbb{C}}\big(\mathbf{1}_{\mathbb{C}} - [L_n] + [\det(\rho_n, L_n)]\big) \in KK_0(\mathbb{K}(F), \mathbb{C}).$$

As a consequence, the following sequence of K-groups is exact:

$$0 \longrightarrow K_1(\mathbb{O}) \xrightarrow{([F]\widehat{\otimes}_{\mathbb{K}(F)}\cdot)\circ\partial} K_0(\mathbb{C}) \xrightarrow{\mathbf{1}_{\mathbb{C}} - [L_n] + [\det(\rho_n, L_n)]} K_0(\mathbb{C}) \xrightarrow{i_*} K_0(\mathbb{O}) \longrightarrow 0$$

The Euler class of an irreducible representation is the alternating sum

$$\begin{aligned}\chi(\rho_n, L_n) &:= \mathbf{1}_{\mathbb{C}} - [L_n] + [\det(\rho_n, L_n)] \in KK(\mathbb{C}, \mathbb{C}) \\ &= (2 - (n+1))\mathbf{1}_{\mathbb{C}}\end{aligned}$$

We have *Gysin*-type exact sequence

$$0 \longrightarrow K_1(\mathbb{O}) \xrightarrow{([F]\widehat{\otimes}_{\mathbb{K}(F)}\cdot)\circ\partial} K_0(\mathbb{C}) \xrightarrow{\mathbf{1}_{\mathbb{C}}-[L_n]+[\det(\rho_n,L_n)]} K_0(\mathbb{C}) \xrightarrow{i_*} K_0(\mathbb{O}) \longrightarrow 0$$

for every $n \in \mathbb{N}$.

We can compute the K-theory groups of the Cuntz–Pimsner algberas of the $SU(2)$-subproduct system arising from $\rho_n : SU(2) \to U(L_n)$:

$$K_0(\mathbb{O}(\rho_n, L_n)) \cong \mathbb{Z}/(n-1)\mathbb{Z} \qquad K_1(\mathbb{O}(\rho_n, L_n)) \cong \begin{cases} \mathbb{Z} & n = 1, \\ \{0\} & \text{otherwise.} \end{cases} \tag{10}$$

- Cuntz–Pimsner algebras are a model for circle bundles.
- Cuntz–Pimsner algebas of subproduct systems are suitable to encode spherical symmetries.
- Open questions:
  - Universal properties;
  - Extend the result to reducible representations;
  - Going from spheres to sphere bundles;