Toposes in arithmetic noncommutative geometry

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Based on joint work in progress with Morgan Rogers and on joint work in progress with Aurélien Sagnier.

Background

In their approach to the Riemann Hypothesis, Connes and Consani introduced in 2014 the Arithmetic Site: the topos $\mathbf{PSh}(\mathbb{N}_+^\times)$ equipped with a sheaf of semirings.

Later, an analogue of the Arithmetic Site was constructed for imaginary quadratic fields with class number 1 (Sagnier, 2017) and for 2×2 integer matrices (H., 2018), the latter without notion of structure sheaf.

In all cases, the involved toposes are toposes of presheaves on monoids. Joint work with Morgan Rogers (2020): what does $\mathbf{PSh}(M)$ look like in terms of M?

Focus of this talk: geometric understanding of "Arithmetic Sites" over different base rings, and the interaction between them. No structures sheaves yet!

Section 1

Short introduction to topos theory

Grothendieck topologies

Definition

Let $\mathcal{C}$ be a category. For U an object of $\mathcal{C}$, a sieve on U is a collection of morphisms $S = \{U_i \to U\}_{i \in I}$ such that $f \in S \Rightarrow f \circ g \in S$.

Grothendieck topologies

Definition

A Grothendieck topology is a collection of sieves (“covering sieves”) on each object, such that

1. the maximal sieve containing all morphisms to U is a covering sieve on U;
2. if S is a covering sieve on U and $f : V \to U$ a morphism, then $f^{-1}S = \{g : f \circ g \in S\}$ is a covering sieve on V;
3. if S is a covering sieve on U and R is a sieve on U such that $f^{-1}R$ is a covering sieve for all $f \in S$, then R is a covering sieve on U.

Sheaves

Definition

A sheaf $\mathcal{F}$ for a Grothendieck topology J on $\mathcal{C}$ is

1. a set $\mathcal{F}(U)$ of "sections" for each object U in $\mathcal{C}$;
2. a restriction function $\rho_f : \mathcal{F}(U) \to \mathcal{F}(V)$ for each $f : V \to U$ such that $\rho_{f \circ g} = \rho_g \circ \rho_f$ and $\rho_1 = 1$;
3. such that for each covering sieve $\{f_i : U_i \to U\}_{i \in I}$ and family of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$, with $\rho_g(s_i) = s_j$, $\forall g : U_j \to U_i$, there is a unique "glued" section $s \in \mathcal{F}(U)$ such that $\rho_{f_i}(s) = s_i$, $\forall i \in I$.

If only 1 and 2 are satisfied, we say that $\mathcal{F}$ is a presheaf.

Sheaves

A morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is a function $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for each U in $\mathcal{C}$, in a way compatible with the restriction functions.

We write $\mathbf{Sh}(\mathcal{C}, J)$ for the category of J-sheaves on $\mathcal{C}$.

Definition

A Grothendieck topos is a category of the form $\mathbf{Sh}(\mathcal{C}, J)$.

If $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{D}, K)$, then we say that $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ are Morita equivalent.

Section 2

Families of toposes

Topological spaces

Let X be a topological space. We write $\mathcal{O}(X)$ for the category of open subsets of X and inclusion maps $V \to U$ whenever $V \subseteq U$.

Let J_{can} be the canonical Grothendieck topology: a sieve $\{U_i \to U\}_{i \in I}$ is a covering sieve if and only if $\bigcup_{i \in I} U_i = U$.

We write $\mathbf{Sh}(X) = \mathbf{Sh}(\mathcal{O}(X), J_{\mathrm{can}})$.

Sheaves on X correspond to local homeomorphisms $Y \to X$.

Equivariant sheaves

Let X be a topological space and let G be a discrete group with a continuous left action on X. Then we define the category $\mathcal{O}_G(X)$ with

- as objects the open sets $U \subseteq X$;
- as morphisms $\mathrm{Hom}(U, V) = \{g \in G : g(U) \subseteq V\}$.
- composition given by multiplication in G.

It has a Grothendieck topology J with as covering sieves those sieves $\{g_i : U_i \to U\}_{i \in I}$ such that $\bigcup_{i \in I} g_i(U_i) = U$.

Now $\mathbf{Sh}_G(X) = \mathbf{Sh}(\mathcal{O}_G(X), J)$ is called the topos of G-equivariant sheaves on X.

Presheaf toposes

Let $\mathcal{C}$ be a category. The presheaf/chaotic topology J_{psh} on $\mathcal{C}$ is the topology such that only the maximal sieves are covering sieves.

The J_{psh}-sheaves are exactly the presheaves.

We write $\mathbf{PSh}(\mathcal{C}) = \mathbf{Sh}(\mathcal{C}, J_{\mathrm{psh}})$ for the category of presheaves on $\mathcal{C}$.

Monoid toposes

Let $\mathcal{C}$ be a category with only one object $*$. We identify $\mathcal{C}$ with the monoid of endomorphisms $M = \mathrm{End}(*)$.

We then write $\mathbf{PSh}(M) = \mathbf{PSh}(\mathcal{C})$.

Example: the underlying topos for the Arithmetic Site of Connes and Consani is $\mathbf{PSh}(\mathbb{N}_+^\times)$, with $\mathbb{N}_+^\times = \{1, 2, 3, \dots\}$ as a monoid under multiplication.

Comparison Lemma

Lemma (Comparison Lemma)

Let $\mathcal{C}$ be a category with Grothendieck topology J. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory such that on each object U of $\mathcal{C}$ there is a J-covering sieve generated by objects in $\mathcal{D}$. Then there is an equivalence $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{D}, J|_{\mathcal{D}})$.

Comparison Lemma

The Comparison Lemma shows that $\mathbf{Sh}(X) \simeq \mathbf{Sh}(\mathcal{B}, J_{\mathrm{can}}|_{\mathcal{B}})$ for a basis of open subsets $\mathcal{B} \subseteq \mathcal{O}(X)$. So sheaves can be defined on a basis.

The same works for G-equivariant sheaves.

Comparison Lemma: Application

Suppose that X is a topological space with continuous left action of a discrete group G.

Assume that the G-action is a covering action, i.e. there is a G-invariant basis $\mathcal{B}$ for X such that $U \cap g(U) \neq \varnothing \Rightarrow g = 1$ for all $U \in \mathcal{B}$.

Then $\mathrm{Hom}(U, V) = \{g \in G : g(U) \subseteq V\}$ has at most one element, for all $U, V \in \mathcal{B}$.

This leads to an equivalence $\mathbf{Sh}_G(X) \simeq \mathbf{Sh}(X/G)$, where X/G has the quotient topology.

For example: $\mathbf{Sh}_{\mathbb{Z}}(\mathbb{R}) = \mathbf{Sh}(S^1)$.

Section 3

Toposes associated to Dedekind domains

(joint work with Aurélien Sagnier)

Lattices

Let R be a Dedekind domain with field of fractions K.

Examples: $\mathcal{O}_K$ for K a number field, the coordinate ring of a smooth, irreducible, affine curve over $\mathbb{F}_q$, and discrete valuations rings like $\mathbb{Z}_p$ or $\mathbb{F}_q[\![t]\!]$.

An R-submodule $M \subseteq K^n$ is said to be full if $MK = K^n$.

Because R is a Dedekind domain, R-submodules $M \subseteq K^n$ are divisorial, i.e.

$$M = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} M_{\mathfrak{p}}.$$

Topological interpretation

We say that $\Lambda \subseteq K^n$ is a lattice if it is full and finitely generated.

We consider the space X_R with

- as points the full R-submodules $M \subseteq K^n$;
- as basis of open sets the subsets $\{M : \Lambda \subseteq M\}$ for Λ a lattice.

This is sober topological space, and as a result we can recover X_R from $\mathbf{Sh}(X_R)$.

Explicit description

We want to give a more explicit description of the full R-submodules $M \subseteq K^n$. We first look at the R-submodules $R^n \subseteq M \subseteq K^n$.

These are equivalently given by a collection $\{M_{\mathfrak{p}}\}_{\mathfrak{p}}$ with $R_{\mathfrak{p}}^n \subseteq M_{\mathfrak{p}} \subseteq K^n$ an $R_{\mathfrak{p}}$-submodule, where $\mathfrak{p}$ ranges over the height one primes (Orzech).

So enough to look at each height one prime separately.

Pontryagin duality

By Morita equivalence, the $R_{\mathfrak{p}}$-submodules $R_{\mathfrak{p}}^n \subseteq M \subseteq K^n$ correspond to the left $\mathrm{M}_n(R_{\mathfrak{p}})$-submodules $\mathrm{M}_n(R_{\mathfrak{p}}) \subseteq M' \subseteq \mathrm{M}_n(K)$, and in turn these correspond to the left $\mathrm{M}_n(R_{\mathfrak{p}})$-submodules of $\mathrm{M}_n(K)/\mathrm{M}_n(R_{\mathfrak{p}})$.

From now on, we assume that $R_{\mathfrak{p}}/\mathfrak{p}$ is finite, for each height one prime $\mathfrak{p}$.

Pontryagin duality

For $\pi \in R_{\mathfrak{p}}$ a uniformizer, note that:

$$\mathrm{M}_n(K)/\mathrm{M}_n(R_{\mathfrak{p}}) = \varinjlim_{k \in \mathbb{N}} (\pi^{-k}\,\mathrm{M}_n(R_{\mathfrak{p}}))/\mathrm{M}_n(R_{\mathfrak{p}}).$$

So the Pontryagin dual of $\mathrm{M}_n(K)/\mathrm{M}_n(R_{\mathfrak{p}})$ is

$$\varprojlim_{k \in \mathbb{N}} \mathrm{M}_n(R_{\mathfrak{p}})/\pi^k \mathrm{M}_n(R_{\mathfrak{p}}) = \mathrm{M}_n(\widehat{R_{\mathfrak{p}}})$$

where

$$\widehat{R_{\mathfrak{p}}} = \varprojlim_{k \in \mathbb{N}} R_{\mathfrak{p}}/\pi^k R_{\mathfrak{p}}$$

is the profinite completion of $R_{\mathfrak{p}}$.

Pontryagin duality

It follows from Pontryagin duality for modules (Martin D. Levin) that $R_\mathfrak{p}$-submodules $M \subseteq K^n/R_\mathfrak{p}^n$ correspond to the closed right $\mathrm{M}_n(R_\mathfrak{p})$-submodules of $\mathrm{M}_n(\widehat{R_\mathfrak{p}})$.

Since $\mathrm{M}_n(R_\mathfrak{p}) \subseteq \mathrm{M}_n(\widehat{R_\mathfrak{p}})$ is dense, these are precisely the closed right ideals of $\mathrm{M}_n(\widehat{R_\mathfrak{p}})$.

But in $\mathrm{M}_n(\widehat{R_\mathfrak{p}})$ every right ideal is principal (Newman–Pierce).

Further, if two elements $a, b \in \mathrm{M}_n(\widehat{R_\mathfrak{p}})$ determine the same principal right ideal, then there is some $u \in \mathrm{GL}_n(\widehat{R_\mathfrak{p}})$ such that $au = b$ (Kaplansky).

Explicit description

So the R-submodules $R^n \subseteq M \subseteq K^n$ are given by the quotient

$$U_{n,R} = \prod_{\mathrm{ht}(\mathfrak{p})=1} \mathrm{M}_n(\widehat{R_\mathfrak{p}})/\mathrm{GL}_n(\widehat{R_\mathfrak{p}}) = \mathrm{M}_n(\widehat{R})/\mathrm{GL}_n(\widehat{R}),$$

where we use the shorthand $\widehat{R} = \prod_{\mathrm{ht}(\mathfrak{p})=1} \widehat{R_\mathfrak{p}}$.

To get all the full R-submodules $M \subseteq K^n$ we have to allow scaling by some $\lambda \in K$, so we get

$$X_{n,R} = \mathrm{M}_n(\mathbb{A}_K^f)/\mathrm{GL}_n(\widehat{R})$$

where $\mathbb{A}_K^f = \widehat{R} \otimes_R K$ is the ring of finite adeles over K.

Isomorphism classes

There is a continuous left $\mathrm{GL}_n(K)$-action on $\mathrm{M}_n(\mathbb{A}_K^f)/\mathrm{GL}_n(\widehat{R})$.

Two points are in the same orbit if and only if the associated R-submodules $M \subseteq K^n$ are isomorphic.

So the topos describing full R-submodules $M \subseteq K^n$ up to isomorphism is

$$\mathcal{E} = \mathbf{Sh}_{\mathrm{GL}_n(K)}(\mathrm{M}_n(\mathbb{A}_K^f)/\mathrm{GL}_n(\widehat{R}))$$

Do we have an alternative description of this topos?

Alternative description

Recall that a basis of open sets is given by the sets

$$\{M : \Lambda \subseteq M\}$$

for lattices Λ. Using this, $\mathcal{E}$ can be described as the topos of sheaves on the category with

- as objects the lattices $\Lambda \subseteq K^n$;
- as morphisms $\Lambda \subseteq \Lambda'$ the elements $g \in \mathrm{GL}_n(K)$ such that $g^{-1}(\Lambda') \subseteq \Lambda$; these are precisely the injective morphisms of R-modules $\Lambda' \to \Lambda$;
- with Grothendieck topology such that a sieve $\{g_i : \Lambda_i \to \Lambda\}$ is a covering sieve if for every $M \subseteq K^n$ of rank n we have: $\Lambda \subseteq M \Rightarrow \exists i,\ g_i(\Lambda_i) \subseteq M$; this is the presheaf topology!

Alternative description

Let $\mathcal{C}_{n,R}$ be the category of (rank n) lattices and injective R-module morphisms between them. We showed that:

$$\mathbf{Sh}_{\mathrm{GL}_n(K)}(\mathrm{M}_n(\mathbb{A}_K^f)/\mathrm{GL}_n(\widehat{R})) \simeq \mathbf{PSh}(\mathcal{C}_{n,R}^{\mathrm{op}}).$$

In the special case that R is a PID, the only rank n lattice up to isomorphism is R^n. In this case, $\mathcal{C}_{n,R}^{\mathrm{op}}$ is (up to equivalence) the monoid

$$\mathrm{M}_n^{\mathrm{ns}}(R) = \{a \in \mathrm{M}_n(R) : \det(a) \neq 0\}.$$

Examples

$$\mathbf{PSh}(\mathbb{Z}^{\mathrm{ns}}) \simeq \mathbf{Sh}_{\mathbb{Q}^*}(\mathbb{A}^f_{\mathbb{Q}}/\widehat{\mathbb{Z}}^*)$$

$$\mathbf{PSh}(\mathrm{M}_n^{\mathrm{ns}}(\mathbb{Z})) \simeq \mathbf{Sh}_{\mathrm{GL}_n(\mathbb{Q})}(\mathrm{M}_n(\mathbb{A}^f_{\mathbb{Q}})/\mathrm{GL}_n(\widehat{\mathbb{Z}}))$$

$$\mathbf{PSh}(\mathbb{Z}[i]^{\mathrm{ns}}) \simeq \mathbf{Sh}_{\mathbb{Q}(i)^*}(\mathbb{A}^f_{\mathbb{Q}(i)}/\widehat{\mathbb{Z}[i]}^*)$$

(Here $R^{\mathrm{ns}} = R - \{0\}$ as a monoid under multiplication.)

Section 4

Change of ring

(joint work with Morgan Rogers)

Geometric morphisms

Definition

For toposes $\mathcal{E}$, $\mathcal{F}$ a geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is given by a functor $f^* : \mathcal{F} \to \mathcal{E}$ (the "inverse image functor") such that f^* preserves colimits and finite limits.

Generalizes the notion of continuous map: if X and Y are sober topological spaces, then the geometric morphisms $\mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ are given by the continuous maps $X \to Y$.

Similarly, the notion of local homeomorphism generalizes to toposes.

Root topos

For M a monoid, let $\int_M M$ be the category of elements of M as a right M-set:

- the objects are the elements of M;
- the morphisms are given by $x \to y$ are given by the elements $m \in M$ such that $ym = x$.

Definition (Connes–Consani, 2019)

The root topos is defined to be the category of presheaves

$$\mathfrak{Root}(M) \simeq \mathbf{PSh}\left(\int_M M\right).$$

$\mathfrak{Root}(M)$ only depends on $\mathbf{PSh}(M)$, not on M itself.

Root topos

There is a local homeomorphism $\mathfrak{Root}(M) \to \mathbf{PSh}(M)$.
We can think of this as a universal covering of $\mathbf{PSh}(M)$.

In the case, $M = \mathbb{N}_+^\times$ this is a Diaconescu cover of the Arithmetic Site given by the big cell (Le Bruyn, 2016).

What if $M = M_n^{\mathrm{ns}}(R)$ for R a PID as above?

Then $\mathfrak{Root}(M) = \mathbf{Sh}(\mathrm{M}_n(\widehat{R})/\mathrm{GL}_n(\widehat{R}))$.

Local homeomorphisms

For M, N cancellative monoids, a local homeomorphism $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is necessarily induced by an injective monoid map $\varphi : M \to N$.

In fact, φ induces a commutative square

$$\begin{array}{ccc} \mathfrak{Root}(M) & \xrightarrow{\mathfrak{Root}(f)} & \mathfrak{Root}(N) \\ {\scriptstyle t_M}\downarrow & & \downarrow{\scriptstyle t_N} \\ \mathbf{PSh}(M) & \xrightarrow{f} & \mathbf{PSh}(N). \end{array}$$

Proposition

The geometric morphism f is a local homeomorphism if and only if $\mathfrak{Root}(f)$ is an equivalence.

Local homeomorphisms

For example the geometric morphisms induced by

- $\mathbb{N}_+^\times \subseteq \mathbb{Z}^{\mathrm{ns}}$
- $R_\mathfrak{p}^{\mathrm{ns}} \subseteq \widehat{R_\mathfrak{p}}^{\mathrm{ns}}$
- any inclusion of groups

are local homeomorphisms.

Local connectedness

Similar result for a larger family of geometric morphisms: the locally connected geometric morphisms.

Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be a geometric morphism induced by a monoid map $\varphi : M \to N$. Then:

Proposition

The geometric morphism f is locally connected if and only if $\mathfrak{Root}(f)$ is locally connected.

Local connectedness

For example the geometric morphisms induced by

- $R^{\mathrm{ns}} \subseteq R[1/f]^{\mathrm{ns}}$
- $R^{\mathrm{ns}} \subseteq R_{\mathfrak{p}}^{\mathrm{ns}}$
- $R^{\mathrm{ns}} \subseteq K^*$

are locally connected.

In the other direction, we can write $\mathbf{PSh}(R[1/f]^{\mathrm{ns}})$, $\mathbf{PSh}(R_{\mathfrak{p}}^{\mathrm{ns}})$ and $\mathbf{PSh}(K^*)$ as subtoposes of $\mathbf{PSh}(R^{\mathrm{ns}})$.