Crossed products and the Atiyah Problem

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Outline

In connection with his work with Singer, Atiyah introduced ℓ^2-Betti numbers.

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ℓ^2-Betti numbers are defined as von Neumann dimensions of homology Hilbert modules of ℓ^2-chain complexes associated to finite, connected, free G-CW-complexes.

In his paper Atiyah observed that, in case G is a finite group, the ℓ^2-Betti numbers coincide, *modulo rescaling by* $|G|$, with the previously-known Betti numbers. Thus in this situation, they are in fact *rational* numbers, and indeed Atiyah posed the following question, which is nowadays commonly known as the *Atiyah Conjecture*.

Question (Atiyah)

Is it possible to obtain irrational values of ℓ^2-Betti numbers?

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Before we give more details, we will introduce some basic definitions.

ℓ^2-Betti numbers for group algebras

Let G be a discrete, countable group. For any subring $R \subseteq \mathbb{C}$ closed under complex conjugation, let

$$RG = \{\sum_{\gamma \in G} r_\gamma \gamma : r_\gamma \in R \text{ almost all } 0\}$$

denote the *group $*$-algebra* of G with coefficients in R.

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Let $\ell^2(G)$ denote the Hilbert space of all square-summable functions $f : G \to \mathbb{C}$, with

$$\langle f, g \rangle_{\ell^2(G)} = \sum_{\gamma \in G} f(\gamma)\overline{g(\gamma)} \quad \text{for } f, g \in \ell^2(G).$$

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The space $\ell^2(G)$ has an orthonormal basis

$$\{\xi_\gamma \in \ell^2(G) : \gamma \in G\}.$$

Observe that G acts faithfully on $\ell^2(G)$ by left (resp. right) multiplication $\lambda : G \to \mathcal{B}(\ell^2(G))$ (resp. $\rho : G \to \mathcal{B}(\ell^2(G))$), defined by

$$\lambda_\gamma(f)(\delta) = f(\gamma^{-1}\delta) \quad \text{and} \quad \rho_\gamma(f)(\delta) = f(\delta\gamma)$$

for $f \in \ell^2(G)$ and $\gamma, \delta \in G$. Both λ and ρ extend R-linearly to actions of RG on $\ell^2(G)$ by bounded operators, preserving the $*$-operation. We will identify RG with the image of λ inside $\mathcal{B}(\ell^2(G))$.

Let

$$\mathcal{N}(G) = \{ T \in \mathcal{B}(\ell^2(G)) : \rho_\gamma \circ T = T \circ \rho_\gamma \ \ \forall \gamma \in G \}$$

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The algebra $\mathcal{N}(G)$ is endowed with a normal, positive and faithful trace, defined as

$$\mathrm{tr}_{\mathcal{N}(G)}(T) := \langle T(\xi_e), \xi_e \rangle_{\ell^2(G)} \quad \text{for } T \in \mathcal{N}(G).$$

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Note that for an element $T = \sum_{\gamma \in G} a_\gamma \gamma \in \mathbb{C}G$, its trace is simply the coefficient a_e.

More generally, $M_k(RG)$ acts faithfully on $\ell^2(G)^k$ by left (resp. right) multiplication, and we have $\mathcal{N}_k(G) = M_k(\mathcal{N}(G))$.

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The trace $\mathrm{tr}_{\mathcal{N}(G)}$ can be extended to an unnormalized trace over $M_k(\mathcal{N}(G))$ by setting

$$\mathrm{Tr}_{\mathcal{N}_k(G)}(T) := \sum_{i=1}^{k} \mathrm{tr}_{\mathcal{N}(G)}(T_{ii})$$

for $T = (T_{ij}) \in M_k(\mathcal{N}(G))$.

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For $V \leq \ell^2(G)^k$ a f.g. Hilbert G-module, the corresponding orthogonal projection operator $p_V : \ell^2(G)^k \to \ell^2(G)^k$ onto V belongs to $\mathcal{N}_k(G)$.

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The *von Neumann dimension of V* is the trace of p_V:

$$\boxed{\dim_{\mathcal{N}(G)}(V) := \mathrm{Tr}_{\mathcal{N}_k(G)}(p_V)}$$

Definition

Let A be a matrix operator in $M_k(\mathbb{C}G)$ for some integer $k \geq 1$. We define the ℓ^2-*Betti number of* A by

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For a subfield $K \subseteq \mathbb{C}$ closed under conjugation, the set of all ℓ^2-Betti numbers of operators $A \in M_k(KG)$ will be denoted by $\mathcal{C}(G, K)$, and will be referred to as the set of all *ℓ^2-Betti numbers arising from G with coefficients in K*.

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$\mathcal{C}(G, K)$ is always a **subsemigroup** of $(\mathbb{R}^+, +)$.
We also write $\mathcal{G}(G, K)$ for the subgroup of $(\mathbb{R}, +)$ generated by $\mathcal{C}(G, K)$.

Following this algebraic point of view, stronger questions –in relation with the original Atiyah question– were formulated over the years. One of its strongest versions is the so-called *Strong Atiyah Conjecture*, first considered by Schick:

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Strong Atiyah Conjecture (SAC) (Schick)

The set of ℓ^2-Betti numbers arising from G with coefficients in K is contained in the subgroup $\sum_H \frac{1}{|H|}\mathbb{Z}$ of $\mathbb{Q}$, where H ranges over the finite subgroups of G.

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The lamplighter group is precisely the first counterexample to the SAC, as proven by R. I. Grigorchuk and A. Żuk.

The bounded case of the SAC (still open!)

Strong Atiyah Conjecture, bounded case (BSAC) (Lück-Schick)

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It is easy to show that, for a torsion-free group, the SAC implies Kaplansky's Conjecture (for $K = \mathbb{C}$). Indeed if $a \in \mathbb{C}G \setminus \{0\}$ then $0 \leq \dim_{\mathcal{N}(G)}(\ker a) < 1$ so that $\dim_{\mathcal{N}(G)}(\ker a) = 0$ and thus $\ker a = \{0\}$ which immediately implies that a is a non-zero divisor.

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Theorem

Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable groups.

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- The class of locally indicable groups is quite large. For instance, it contains all torsion-free one-relator groups.

Theorem

(Jaikin-Zapirain, López-Álvarez) Let G be a locally indicable group. Then G satisfies the SAC.

Definition

The lamplighter group L is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$:

$$L = \mathbb{Z}_2 \wr \mathbb{Z} = \Big(\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_2 \Big) \rtimes_\sigma \mathbb{Z}$$

where the semidirect product is taken with respect to the Bernoulli shift $\sigma : \mathbb{Z} \curvearrowright \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_2$:

$$\sigma(x)_i = x_{i+1} \quad \text{for } x = (x_i) \in \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_2.$$

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In terms of generators and relations, if we denote by t the generator corresponding to $\mathbb{Z}$, and by a_i the generator corresponding to the i^{th} copy of $\mathbb{Z}_2$, we have the presentation:

The non-bounded case. Counterexamples to the SAC.

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$$\begin{aligned} L &= \langle t, \{a_i\}_{i \in \mathbb{Z}} \mid a_i^2,\ a_i a_j a_i a_j,\ t a_i t^{-1} a_{i-1} \text{ for } i, j \in \mathbb{Z} \rangle \\ &= \langle t, a_0 \mid a_0^2, [a_0, t^i a_0 t^{-i}], i \in \mathbb{N} \rangle. \end{aligned}$$

Theorem (Grigorchuk-Żuk, 2001)

The lamplighter group L is a counterexample to the SAC. Indeed, all finite subgroups of L have orders of the form 2^n, but the element

$$a = e_0 t + t^{-1} e_0,$$

where $e_0 = \frac{1}{2}(1 + a_0)$ *satisfies that* $\dim_{\mathcal{N}(L)}(\ker a) = \frac{1}{3}$.

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A simpler proof was later obtained by Dicks and Schick.
This shows that the lamplighter group is a counterexample to the SAC. But what about the original Atiyah's problem?

In 2013, Austin showed that there exist irrational ℓ^2-Betti numbers by showing that the set of all ℓ^2-Betti numbers (for all groups) is an uncountable subset of $\mathbb{R}$.

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Indeed, Austin considered ℓ^2-Betti numbers coming from an uncountable family of groups of the form $\Gamma \rtimes \mathbb{F}_2$, where Γ is an abelian group, and $\mathbb{F}_2$ is the free group on 2 elements, acting on Γ.

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$$\Big(\lambda\colon \mathbb{C}(U \rtimes_\alpha \Lambda) \subseteq \mathcal{N}(U \rtimes_\alpha \Lambda) \curvearrowright \ell^2(U \rtimes_\alpha \Lambda)\Big) \xrightarrow{\cong}$$

$$\Big(\pi\colon \mathbb{C}(U \rtimes_\alpha \Lambda) \subseteq L^\infty(m_{\hat{U}}) \rtimes_{\hat{\alpha}} \Lambda \curvearrowright L^2(m_{\hat{U}} \otimes \#_\Lambda)\Big)$$

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Here $m_{\hat{U}}$ is the normalized Haar measure on the compact dual group $\hat{U}$ of U, $\#_\Lambda$ is the counting measure on Λ, and π is the left regular representation of the group measure von Neumann algebra $L^\infty(m_{\hat{U}}) \rtimes_{\hat{\alpha}} \Lambda$ on $L^2(m_{\hat{U}} \otimes \#_\Lambda)$.

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Theorem (Grabowski, 2014)

The set of ℓ^2-Betti numbers arising from L^3 contains the irrational number:

$$\frac{1}{64} - \frac{1}{8}\sum_{k=1}^{\infty}\frac{1}{2^{k^2+4k+6}}.$$

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Grabowski also stated a particular form of the Atiyah Problem:

Question (Atiyah Problem for G)

Given a group G, what is the set of ℓ^2-Betti numbers arising from G?

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Theorem (Grabowski, 2016)

There exists a matrix T with entries in $\mathbb{Z}L_p = \mathbb{Z}(\mathbb{Z}_p \wr \mathbb{Z})$ such that

$$\dim(\ker T) = 1344\Big(\frac{4p^3+3p^2+2p-1}{8p^3} + \frac{1}{8p^3}\sum_{k=1}^{\infty}\Big(\frac{p-1}{p}\Big)^{k+2^k}\Big)$$

$*$-regular rings and Sylvester matrix rank functions

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Definition

A *(von Neumann) regular ring* is a ring R such that for each $x \in R$ there is $y \in R$ such that $x = xyx$. A *$*$-regular ring* is a regular ring R endowed with a *proper* involution $*$, that is, $x^*x = 0$ if and only if $x = 0$.

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This approach is fundamental in the work of Jaikin-Zapirain and López-Álvarez, as well as in my joint work with Joan Claramunt.

Definition

A *(von Neumann) regular ring* is a ring R such that for each $x \in R$ there is $y \in R$ such that $x = xyx$. A *$*$-regular ring* is a regular ring R endowed with a *proper* involution $*$, that is, $x^*x = 0$ if and only if $x = 0$.

Example (Murray-von Neumann)

Let $\mathcal{N}$ be a *finite* von Neumann algebra in $\mathcal{B}(\mathcal{H})$. Then the algebra of unbounded closed operators affiliated to $\mathcal{N}$ is a $*$-regular ring. We denote by $\mathcal{U}(G)$ the *-algebra of unbounded closed operators affiliated to $\mathcal{N}(G)$.

In a $*$-regular ring R, for every $x \in R$ there exist unique projections $e, f \in R$ such that $xR = eR$ and $Rx = Rf$. It is common to denote them by $e = \mathrm{LP}(x)$ and $f = \mathrm{RP}(x)$, and are termed the *left* and *right projections* of x, respectively.

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Definition

For any subset $S \subseteq R$ of a unital $*$-regular ring, there exists a smallest unital $*$-regular subring of R containing S. This $*$-regular ring is denoted by $\mathcal{R}(S, R)$, and called the *$*$-regular closure* of S in R.

Let us denote by $M(R)$ the set of finite matrices over R of arbitrary size, i.e. $M(R) = \bigcup_{n \geq 1} M_n(R)$.

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The notion of Sylvester matrix rank function was first introduced by Malcolmson in order to characterize homomorphisms from a fixed ring to division rings.

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The ring operations are continuous with respect to d, and rk extends uniquely to a Sylvester matrix rank function $\overline{\mathrm{rk}}$ on the completion $\overline{R}$ of R with respect to d.

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$$\mathrm{rk}_{\mathcal{U}(G)}(U) := \mathrm{Tr}_{\mathcal{N}_k(G)}(\mathrm{LP}(U)) = \mathrm{Tr}_{\mathcal{N}_k(G)}(\mathrm{RP}(U))$$

for any matrix $U \in M_k(\mathcal{U}(G))$, where $\mathrm{LP}(U)$ and $\mathrm{RP}(U)$ are the left and right support projections of U inside the $*$-regular algebra $M_k(\mathcal{U}(G))$ respectively (which actually belong to $\mathcal{N}_k(G)$).

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In particular, we obtain by restriction a Sylvester matrix rank function rk_{KG} over KG. So for a matrix operator $A \in M_k(KG)$ we have $p_{\ker A} = 1_k - \mathrm{RP}(A)$ and we get the equality

$$b^{(2)}(A) := \dim_{\mathcal{N}(G)}(\ker A) = k - \mathrm{rk}_{KG}(A).$$

Here 1_k stands for the identity matrix in k dimensions.

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$$\mathcal{G}_{\mathrm{rk}}(R) = \mathcal{C}_{\mathrm{rk}}(R) - \mathcal{C}_{\mathrm{rk}}(R) = \mathcal{C}'_{\mathrm{rk}}(R) - \mathcal{C}'_{\mathrm{rk}}(R).$$

The following is an immediate corollary of a result by Jaikin-Zapirain:

Theorem

If $\mathcal{U}$ is a $*$-regular ring and R is a $*$-subring of $\mathcal{U}$ such that $\mathcal{U}$ is the $*$-regular closure of R in $\mathcal{U}$, then we have

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- Under which conditions does the equality $\mathcal{C}_{\mathrm{rk}}(R) = \mathcal{C}'_{\mathrm{rk}}(R)$ hold?
- Under which conditions $\mathcal{C}_{\mathrm{rk}}(R) = \mathcal{G}_{\mathrm{rk}}(R) \cap \mathbb{R}^+$, or $\mathcal{C}'_{\mathrm{rk}}(R) = \mathcal{G}_{\mathrm{rk}}(R) \cap \mathbb{R}^+$? What about group algebras KG?

Question (Atiyah Problem for rank functions)

Given a pair (R, rk), what is the group $\mathcal{G}_{\mathrm{rk}}(R)$? What about the semigroups $\mathcal{C}_{\mathrm{rk}}(R)$ and $\mathcal{C}'_{\mathrm{rk}}(R)$?

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In the above setting, the most natural object to study is the group $\mathcal{G}_{\mathrm{rk}}(R)$, by Jaikin-Zapirain's result.

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The SAC holds for KG $\iff \mathcal{C}(G,K) = \mathbb{Z}^+ \iff \mathcal{G}(G,K) = \mathbb{Z}$ $\iff \phi_{\mathrm{rk}}(K_0(\mathcal{R}_{KG})) = \mathbb{Z} \iff \mathcal{R}_{KG}$ is a division ring.

If G is a discrete group and K is a subfield of $\mathbb{C}$ closed under conjugation, then we denote by $\mathcal{R}_{KG}$ the $*$-regular closure of KG in $\mathcal{U}(G)$.

Observe that $(\mathcal{R}_{KG}, \mathrm{incl}, \mathrm{rk})$ is the epic $*$-regular envelope of (KG, rk).

Using the above we can immediately recover the following well-known result:
Let G be a torsion-free group.

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$\iff \phi_{\mathrm{rk}}(K_0(\mathcal{R}_{KG})) = \mathbb{Z} \iff \mathcal{R}_{KG}$ is a division ring.
Indeed the SAC holds for KG iff there is a division ring $\mathcal{D}$ with

$$KG \subseteq \mathcal{D} \subseteq \mathcal{U}(G)$$

Crossed products and the Atiyah Problem

In my joint work with Joan Claramunt, we use a general approximation construction for $\mathbb{Z}$-crossed product algebras, which is inspired by the work of Putnam et al.

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We consider the algebra $C_K(X)$ of continuous functions $f : X \to K$, where K is endowed with the discrete topology. (So these are the locally constant functions on X)
Now consider the crossed product $*$-algebra:

$$\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$$

Theorem (A-Claramunt, 2020)

There exists a unique faithful Sylvester matrix rank function rk *on* $\mathcal{A}$ *such that*

$$\mathrm{rk}(\chi_U) = \mu(U)$$

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The motivating examples

Suppose now that H is a discrete countable abelian torsion group and that $\mathbb{Z}$ acts on H by automorphisms via $\rho : \mathbb{Z} \curvearrowright H$.

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Proposition (A-Claramunt)

If rk_{KG} *is extremal in* $\mathbb{P}(KG)$ *then* $\hat{\mu}$ *is ergodic, and when applying the above construction to* $\hat{\mu}$ *we end up with a Sylvester matrix rank function* $\mathrm{rk}_{\mathcal{A}}$ *on* $\mathcal{A} = C_K(\hat{H}) \rtimes_T \mathbb{Z}$ *such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{A} & \hookrightarrow & \mathcal{R}_{\mathcal{A}} & \hookrightarrow & \mathfrak{R}_{\mathrm{rk}} \\ \downarrow \cong & & \downarrow \cong & & \downarrow \\ KG & \hookrightarrow & \mathcal{R}_{KG} & \hookrightarrow & \mathcal{U}(G), \end{array}$$

and both rank functions coincide under this isomorphism.

Applying the above to the lamplighter $L = \mathbb{Z}_2 \wr \mathbb{Z}$, we are led to study the algebra

$$KL \cong \mathcal{A} = C_K(X) \rtimes \mathbb{Z},$$

where $X = \{0,1\}^{\mathbb{Z}}$, $T\colon X \to X$ is the shift map $T(x)_i = x_{i+1}$ for $x \in X$ and μ is the product measure of the $(\frac{1}{2}, \frac{1}{2})$-measure on $\{0,1\}$.

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We study the algebra $\mathcal{A}$ and its $*$-regular closure $\mathcal{R}_{\mathcal{A}}$ by using a nested sequence of $*$-subalgebras:

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}$$

which 'almost' cover $\mathcal{A}$.

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where, e.g.

$$[1\underline{0}] = \{(x_i) \in X : x_{-1} = 1, x_0 = 0\}.$$

Theorem (A-Goodearl, 2017)

We have $\mathcal{G}_{\mathsf{rk}}(\mathcal{A}_0) = \mathbb{Q}$.

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Moreover a complete description of the $*$-regular closure of $\mathcal{A}_0$ is done in the paper with Goodearl.

Theorem (A-Claramunt, 2020)

Fix n a non-negative integer. For $0 \le l \le n$, take

$$p_l(x) = a_{0,l} + a_{1,l}x + \cdots + a_{m_l,l}x^{m_l}$$

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Theorem (A-Claramunt, 2020)

Take $K = \mathbb{Q}$. Then $\mathcal{G}_{\mathrm{rk}}(\mathcal{A}_1)$ contains algebraic irrational numbers. In particular:

$$\frac{1}{4}\sqrt{\frac{2}{7}} \in \mathcal{G}_{\mathrm{rk}}(\mathcal{A}_1) \subseteq \mathcal{G}(L, \mathbb{Q}).$$

The odometer algebra

Consider the topological space $X = \{0,1\}^{\mathbb{N}}$ (a Cantor space). Let T be the homeomorphism on X given by the odometer, namely for $x = (x_i) \in X$, T is given by

$$T(x) = \begin{cases} (1, x_2, x_3, \ldots) & \text{if } x_1 = 0, \\ (0, \ldots, 0, 1, x_{m+2}, \ldots) & \text{if } x_j = 1 \text{ for } 1 \leq j \leq m \\ & \text{and } x_{m+1} = 0, \\ (0, 0, \ldots) & \text{if } x_i = 1 \text{ for all } i \in \mathbb{N}. \end{cases}$$

Note that the odometer action is just addition of $(1, 0, \ldots)$ by carry-over.

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and we can think of the above theorem as giving an affirmative answer to the SAC for the algebra $K\mathbb{G}(2) \rtimes_\rho \mathbb{Z}$.

Thank you for your attention!!

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Definition

Let $\mathcal{D}$ be the smallest nonempty class of groups that

• contains every torsion-free group G for which there exists an epimorphism $p\colon G \to A$ onto an elementary amenable group A such that $p^{-1}(H) \in \mathcal{D}$ for all finite subgroup H of A.

• is closed under taking limits, colimits and subgroups.

Theorem

The BSAC holds the following classes of groups:

- *Groups in $\mathcal{D}$*
- *Artin braid groups*
- *Finite extensions of the fundamental group of a compact special cube complex*
- *Torsion-free p-adic analytic pro-p-groups*

see for instance the recent survey paper by Jaikin-Zapirain