Relating Morita equivalence in algebra and geometry via deformation quantization

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Global NCG seminar, 2021

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$\rightsquigarrow$ Role in NC geometry (Rieffel's strong Morita equiv., groupoids...)

◇ Morita equivalence in Poisson geometry (Xu, 1991)

Is there a concrete link via quantization?

Poisson structures $\rightsquigarrow$ noncommutative algebras

Goals:

- Morita equivalence of Poisson structures, extension to formal setting *(B., Ortiz, Waldmann, IMRN 2021; Arxiv:2006.10240)*
- Relation with noncommutative algebras via deformation quantization *(B., Waldmann, Dolgushev, Crelle's J. 2012)*

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Outline:

1. Morita equivalence reminder I: algebras.

   Interlude: elements of Poisson geometry
2. Morita equivalence reminder II: Poisson structures.
3. The bridge: deformation quantization
4. Morita equivalence of formal Poisson structures
5. Relation with Morita equivalence of star products

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- isomorphism implies Morita equivalence
- $\mathcal{A}$ is Morita equivalent to $M_n(\mathcal{A})$ via $\mathcal{A}^n$
- Many natural invariants (e.g. isomorphic centers).
- Group of (iso. classes) self-Morita equivalences: $\mathrm{Pic}(\mathcal{A})$.

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Remarks:

- Morita equivalence has been refined to various types of algebras (e.g. C^*-algebras...)
- Geometric versions of Morita equivalence (for topological/Lie groupoids, Poisson manifolds...)

Interlude: Poisson manifolds

A *Poisson bracket* on P is $\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$:

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Poisson maps: $\varphi : P \to Q$ such that $\varphi^* : C^\infty(Q) \to C^\infty(P)$ preserves brackets.

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$\diamond$ General structure:

Poisson structures $\rightleftharpoons$ (singular) symplectic foliations

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◇ Global versions are *symplectic groupoids*:

$\mathcal{G} \rightrightarrows P, \;\; \omega \in \Omega^2(\mathcal{G})$ symplectic,

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Examples: $\quad T^*P \rightrightarrows P, \quad S \times \bar{S} \rightrightarrows S, \quad T^*G \rightrightarrows \mathfrak{g}^*$.

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$$\begin{array}{ccc} & S & \\ {\scriptstyle J_1} \swarrow & & \searrow {\scriptstyle J_2} \\ P_1 & & P_2, \end{array}$$

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◇ Equivalence bimodules: dual pairs plus "regularity" conditions:

- Maps are surjective submersions, 1-connected fibres
- Symplectic orthogonal of each other

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Some examples:

- symplectic manifolds ($\pi_1(S)$), fibrations...
- "Topologically stable" Poisson surfaces (aka Radko surfaces) (B.-Radko, B.-Weinstein), log-symplectic structures...

B-field/gauge transform on Poisson structures and bimodules

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Given $B \in \Omega^2_{cl}(P)$ such that $1 + B\pi : T^*P \to T^*P$ invertible, then

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For bimodules (B., Radko): if $(P_1, \pi_1) \stackrel{J_1}{\leftarrow} (S, \omega) \stackrel{J_2}{\rightarrow} (P_2, \pi_2)$ bimodule, then so is

$$(P_1, \tau_B(\pi_1)) \xleftarrow{J_1} (S, \omega + J_1^* B) \xrightarrow{J_2} (P_2, \pi_2).$$

In particular, any integrable π is Morita equivalent to $\tau_B(\pi)$.

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Concrete link between algebraic and Poisson Morita equivalences?

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Moduli of star products: $\mathrm{Def}(P) = \{\star\}/\sim$

Existence of star products (given Poisson bracket)? Classification?

Formal Poisson structures: $\pi_\hbar \in \hbar\mathcal{X}^2(P)[[\hbar]]$,

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Consequence of Kontsevich's formality ('97):

Theorem: There is 1-1 correspondence

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such that $\frac{1}{\hbar}[f,g]_\star\big|_{\hbar=0} = \pi_1(df, dg)$.

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$$\pi_1(df, dg) = \frac{1}{\hbar}[f, g]_\star\Big|_{\hbar=0}$$

But π_1 is preserved by Morita equivalence of $\star$...

The Problem

Link algebraic and Poisson Morita equivalences via $\mathcal{K}_*$

formal Poisson structures	$\rightleftharpoons$	star products
$\hbar\pi_1 + \hbar^2\pi_2 + \dots$		$(C^\infty(P)[[\hbar]], \star)$

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More interesting to compare Morita equivalences once one makes sense of Morita equivalence of **formal** Poisson structures...

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Equivalence of formal Poisson structures: $\pi = \exp(\mathcal{L}_X)\pi'$.

$$\mathrm{FPois}_{\pi_0} = \{\pi = \pi_0 + \hbar\pi_1 + \ldots, \; |; [\pi, \pi] = 0\,\}/\!\sim$$

Example: Formal symplectic structures:

$\omega = \omega_0 + \sum_{r=1}^{\infty} \hbar^r \omega_r \in \Omega^2_{cl}(P)[[\hbar]]$ such that ω_0 is symplectic.

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Hence

$$\pi_0 \text{ symplectic} \implies \mathrm{FPois}_{\pi_0}(P) = \hbar H^2(P)[[\hbar]].$$

Equivalence bimodules for formal Poisson structures

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such that the underlying *"zeroth-order bimodule"* ($\hbar = 0$):

$$\begin{array}{ccccc} & & (S, \omega_0) & & \\ & \swarrow^{J_1} & & \searrow^{J_2} & \\ (P_1, \pi_0^{(1)}) & & & & (P_2, \pi_0^{(2)}) \end{array}$$

is Morita equivalence in ordinary sense. (here $\Phi^{(i)} = J_i^* + \hbar\phi_1^{(i)} \ldots,$)

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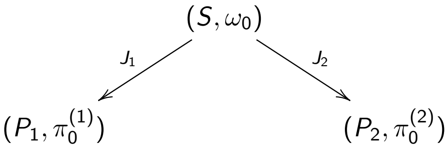

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Can we turn a bimodule $(P_1, \pi_0) \stackrel{J_1}{\leftarrow} (S, \omega_0) \stackrel{J_2}{\rightarrow} (P_2, \pi_0')$ into

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One can produce examples of Morita equivalence with B-fields...

B-fields (formal setting)

Let $\pi = \pi_0 + \hbar\pi_1 + \ldots$ be formal Poisson and $B \in \Omega^2_{cl}(P)[[\hbar]]$.

If $(1 + B_0\pi_0)$ invertible, then

$$\tau_B(\pi) := \pi(1 + B\pi)^{-1}$$

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we obtain new bimodule

$$(C^\infty(P_1)[[\hbar]], \tau_B(\pi^{(1)})) \xrightarrow{\Phi^{(1)}} (C^\infty(S)[[\hbar]], \omega_B) \xleftarrow{\Phi^{(2)}} (C^\infty(P_2)[[\hbar]], \pi^{(2)}),$$

where $\omega_B = \omega + \exp(\mathcal{L}_{Z_{(1)}})(J_1^* B)$, and $\Phi^{(1)} = \exp(\mathcal{L}_{Z_{(1)}})J_1^*$.

Formal Poisson structures vanishing in zeroth order

- We will describe Morita equivalence of formal Poisson structures *vanishing in zeroth order*: $\pi = \hbar\pi_1 + \dots$.
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$$\tau : H^2(P)[[\hbar]] \times \mathrm{FPois}_0(P) \to \mathrm{FPois}_0(P), \quad [\pi] \mapsto [\tau_B(\pi)].$$

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Theorem: *Two formal Poisson structures* π *and* π' *on* P*, vanishing in zeroth order, are Morita equivalent iff there exists* $\psi \in \mathrm{Diff}(P)$ *such that* $[\pi]$ *and* $[\psi_* \pi']$ *lie in the same orbit of* B*-field action on* $\mathrm{FPois}_0(P)$*:* $[\psi_* \pi'] = [\tau_B(\pi)]$.

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Morita equivalence on $\mathrm{FPois}_0(P)$ $=$ $(\mathrm{Diff}(P) \ltimes H^2(P)[[\hbar]])$-orbits

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We must study equivalence bimodules:

$$(C^\infty(P)[[\hbar]], \pi) \xrightarrow{\Phi} (C^\infty(S)[[\hbar]], \omega) \xleftarrow{\Phi'} (C^\infty(P)[[\hbar]], \pi'),$$

- $\pi_0 = 0$, $\pi'_0 = 0$, $\omega = \omega_0 + \hbar\omega_1 + \ldots$.
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Since it is self-equivalence of $(P, 0)$, we have

$$S = T^*P, \quad \omega_0 = \omega_{can} + p^*B_0, \quad J = p : T^*P \to P.$$

Step 2:

For $\pi = \hbar\pi_1 + \dots$ and $B \in \Omega^2_{cl}(P)[\hbar]]$, can define bimodule

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Step 3: Identification of Υ with B-fields.

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Problem (b): Given Φ, let $\mathbf{C} = \Phi(\mathcal{A}[[\hbar]])^c \subset \mathcal{B}[[\hbar]]$. Can we find $Z' \in \hbar \mathcal{X}(S)[[\hbar]]$ such that $\Phi' = \exp(\mathcal{L}_{Z'})\phi'_0$ maps $\mathcal{A}'[[\hbar]]$ to $\mathbf{C}$? Unique?

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Problems (a) and (b) admit ("unique") solutions if

$$H^1_{CE,der}(\mathcal{A}, \mathcal{B}) = H^2_{CE,der}(\mathcal{A}, \mathcal{B}) = 0.$$

5. Morita equivalence of star products

Let $\star_1$ and $\star_2$ be star products on P.

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Morita equivalence on $\mathrm{Def}(P) \;=\; (\mathrm{Diff}(P) \ltimes H^2(P, \mathbb{Z}))$-orbits.

Recall

$$\mathcal{K}_* : \mathrm{FPois}_0(P) \xrightarrow{\sim} \mathrm{Def}(P).$$

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Theorem (B., Dolgushev, Waldmann) If $[\star] = \mathcal{K}_*([\pi])$, then

$$\Psi_L([\star]) = \mathcal{K}_*([\tau_B(\pi)]), \quad \text{where } B = 2\pi i c_1(L)$$

Recall

$$\mathcal{K}_* : \mathrm{FPois}_0(P) \overset{\sim}{\to} \mathrm{Def}(P).$$

How to describe $\mathrm{Pic}(P)$*-action on* $\mathrm{FPois}_0(P)$*?*

Theorem (B., Dolgushev, Waldmann) If $[\star] = \mathcal{K}_*([\pi])$, then

$$\Psi_L([\star]) = \mathcal{K}_*([\tau_B(\pi)]), \quad \text{where } B = 2\pi i c_1(L)$$

Conclusion: Morita equivalence of formal Poisson structures by $(2\pi i)$ *integral* B-fields $B = B_0$ quantize to Morita equivalent star-product algebras (defined by "pre-quantum line bundle").

Thanks for your attention!