Differential Geometry of Weightings

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Global NCG Seminar
February 19, 2021 (online)

Plan:

1. Weightings
2. Weighted normal bundle
3. Weighted tubular neighborhoods
4. Lie filtrations
5. Other directions

Based on:

- Yiannis Loizides, E.M.: Differential geometry of weightings, arXiv:2010.01643
- E.M.: Euler-like vector fields, normal forms, and isotropic embeddings, Indag. Math. **32** (2021) 224-245

1. Weightings

Motivations:

- Normal form results 'with weights'.
- 'Weighted blow-ups' and related concepts.
- Theory of hypo-elliptic differential operators.

E.g.: Kolmogorov operator

$$\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} - \frac{\partial^2}{\partial x^2}.$$

(see talk by Iakovos Androulidakis, December 2020.)

Weightings

Fix a weight sequence of order r

$$0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq r, \quad w_i \in \mathbb{Z}.$$

Assign weight w_a to coordinate function x_a.

Definition

An order r weighting of M along a submanifold N is a filtration (compatible with products)

$$(*) \qquad C^\infty(M) = C^\infty(M)_{(0)} \supseteq C^\infty(M)_{(1)} \supseteq \cdots$$

with $C^\infty(M)_{(1)} = \mathcal{I}_N$, s.t. in suitable local coordinates on $U \subseteq M$,

$$C^\infty(U)_{(i)} = \langle x_1^{s_1} \cdots x_n^{s_n} \colon \sum s_a w_a \geq i \rangle.$$

Equivalent to *quasi-homogeneous structures* (Melrose 1996, Behr 2021).

Weightings

Examples

- $r = 1$ weightings are *trivial*: $C^\infty(M)_{(k)} = (\mathcal{I}_N)^k$.
- $r = 2$ weightings $\Leftrightarrow$ subbundles $F \subseteq \nu(M,N) = TM|_M/TN$. Here filtration is generated by

$$C^\infty(M)_{(1)} = \{f \colon f|_N = 0\},$$
$$C^\infty(M)_{(2)} = \{f \colon f|_N = 0,\ \mathrm{d}f|_F = 0\}$$
$$\ldots\ldots$$

- Any nested sequence of submanifolds

$$N = N_0 \subseteq N_{-1} \subseteq N_{-2} \subseteq \cdots \subseteq N_{-r} = M$$

determines order r weighting.

Weightings

In general, weightings of order r

$$C^\infty(M) = C^\infty(M)_{(0)} \supseteq C^\infty(M)_{(1)} \supseteq \cdots$$

determine a filtration

$$N = F_0 \subseteq F_{-1} \subseteq \cdots \subseteq F_{-r} = \nu(M, N)$$

but are *not* determined by this filtration if $r > 2$.

Question: How to describe weightings in coordinate-free way?

Weightings

Answer: Using jets. Consider the *r-th tangent bundle*

$$T_r M = J_0^r(\mathbb{R}, M) \to M$$

Each $f \in C^\infty(M)$ has lifts $f^{(0)}, \ldots, f^{(r)} \in C^\infty(T_r M)$:

$$f^{(i)}(j^r(\gamma)) = \frac{1}{i!} \frac{d^i}{dt^i}\Big|_{t=0} f(\gamma(t)).$$

Theorem

An order r weighting along N determines, and is determined by, a subbundle

$$\begin{array}{ccc} Q & \longrightarrow & T_r M \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \end{array}$$

given as the vanishing set of all $f^{(i)}$ with $f \in C^\infty(M)_{(i+1)}$.

Weightings

Vector fields $X \in \mathfrak{X}(M)$ have lifts $X^{(0)}, \ldots, X^{(-r)} \in \mathfrak{X}(T_r M)$ defined by

$$\frac{1}{\ell!} X^{(-i)} f^{(\ell)} = \frac{1}{(\ell - i)!} (Xf)^{(\ell - i)}.$$

These span a graded Lie algebra $\mathfrak{g} \subseteq \mathfrak{X}(T_r M)$.

Theorem

A subbundle

$$\begin{array}{ccc} Q & \longrightarrow & T_r M \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \end{array}$$

defines a weighting if and only if

1. *Q invariant under action of $J_{0,0}(\mathbb{R}, \mathbb{R})$ ('reparametrizations').*
2. *Q is an orbit of a graded Lie subalgebra of $\mathfrak{g}$.*

Weightings

The construction is functorial:

Given manifold pairs (M, N), (M', N') with weightings, a smooth map $\Phi\colon M \to M'$ preserves weightings if and only if

$$T_r\Phi\colon (T_r M, Q) \to (T_r M', Q').$$

This allows to formulate compatibilities with extra structure, e.g.:

Definition

A weighting on a Lie groupoid $G \rightrightarrows M$ along a subgroupoid is multiplicative if $Q \subseteq T_r G$ is a subgroupoid.

2. Weighted normal bundle

Weighted normal bundle

Theorem

A weighting along $N \subseteq M$ determines a unique fiber bundle

$$\nu_{\mathcal{W}}(M,N) \to N,$$

with an action $t \mapsto \kappa_t$ of $(\mathbb{R},\cdot)$, such that

$$C^\infty(M)_{(k)}/C^\infty(M)_{(k+1)}$$

are the homogeneous functions of degree k on $\nu_{\mathcal{W}}(M,N)$.

We call this the weighted normal bundle. In short:

$$\nu_{\mathcal{W}}(M,N) = \mathrm{Hom}_{\mathrm{alg}}(\mathrm{gr}(C^\infty(M)),\mathbb{R}).$$

Weighted normal bundle

Remarks:

- $\nu_{\mathcal{W}}(M,N)$ is not a vector bundle, but is a graded fiber bundle (Grabowski-Rodkievicz): a manifold E with $(\mathbb{R},\cdot)$-action.

- There's a functor:

$$\{\text{graded fiber bundles}\} \longrightarrow \{\text{graded vector bundles}\},$$

$$E \mapsto E_{\text{lin}} := \nu(E,N).$$

  Here: $\nu_{\mathcal{W}}(M,N)_{\text{lin}} = \text{gr}\left(\nu(M,N)\right)$.

- In terms of $Q \subseteq T_r M$,

$$\nu_{\mathcal{W}}(M,N) = Q/\sim$$

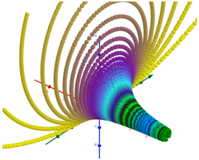
  generalizing $\nu(M,N) = TM|_N/TN$.

Weighted normal bundle

Also have weighted deformation space

$$\mathcal{D}_{\mathcal{W}}(M,N) = \nu_{\mathcal{W}}(M,N) \sqcup (M \times \mathbb{R}^\times) \xrightarrow{\pi} \mathbb{R},$$

with zoom action of $\mathbb{R}^\times$:
$\lambda \cdot v = \lambda v, \qquad \lambda \cdot (m,t) = (m, \lambda^{-1} t)$



- Algebraically:

$$\mathcal{D}_{\mathcal{W}}(M,N) = \mathrm{Hom}_{\mathrm{alg}}(\mathrm{Rees}(C^\infty(M)), \mathbb{R}).$$

- Geometrically: homogeneous extensions

$$C^\infty(M)_{(i)} \to C^\infty(\mathcal{D}_{\mathcal{W}}(M,N))_{[i]}, \quad f \mapsto \widetilde{f}_{[i]}.$$

I.e., $f = \widetilde{f}_{[i]}|_{\pi^{-1}(1)} \in C^\infty(M)$. Call $f_{[i]} = \widetilde{f}_{[i]}|_{\pi^{-1}(0)} \in C^\infty(\nu(M,N))$ the homogeneous approximation.

Weighted normal bundle

Likewise for all tensor fields, e.g. vector fields:

$$X \in \mathfrak{X}(M)_{(i)} \rightsquigarrow \widetilde{X}_{[i]} \in \mathfrak{X}(\mathcal{D}_{\mathcal{W}}(M,N))_{[i]} \rightsquigarrow X_{[i]} \in \mathfrak{X}(\nu_{\mathcal{W}}(M,N))_{[i]}.$$

Example

Let (M,ω) symplectic, $N \subseteq M$ isotropic: $\iota_N^*\omega = 0$. Then

$$TN^{\omega}/TN \subseteq \nu(M,N)$$

determines a weighting of order $r = 2$. One finds that ω has filtration degree 2, and the homogeneous approximation

$$\omega_{[2]} \in \Omega(\nu_{\mathcal{W}}(M,N))_{[2]}$$

is symplectic.

Weighted normal bundle

Given a weighting along $N \subseteq M$ one defines *weighted blow-up* as

$$\big(\mathcal{D}_{\mathcal{W}}(M,N) - N \times \mathbb{R}\big)/\mathbb{R}_{>0} = (M-N) \sqcup \underbrace{S(\nu_{\mathcal{W}}(M,N)) \sqcup (M-N)}_{\mathrm{Bl}_{\mathcal{W}}(M,N)}$$

See Debord-Skandalis for unweighted case; see also 'quasi-homogeneous blow-ups' of Melrose, Behr.

3. Weighted tubular neighborhoods

Weighted tubular neighborhoods

The $(\mathbb{R}, \cdot)$-action on $\nu_{\mathcal{W}}(M, N)$ defines an *Euler vector field*

$$\mathcal{E} \in \mathfrak{X}(\nu_{\mathcal{W}}(M, N)).$$

In adapted coordinates x_i of weights w_i, $\mathcal{E} = \sum_{i=1}^{n} w_i x_i \frac{\partial}{\partial x_i}$.

Definition

$X \in \mathfrak{X}(M)$ is weighted Euler-like if it has filtration degree 0, with weighted homogeneous approximation

$$X_{[0]} = \mathcal{E}.$$

Example ($\mathbb{R}^2$ with weights $1, 2$)

$(x + y^2)\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$ is weighted Euler-like, but

$x\frac{\partial}{\partial x} + (2y + x^2)\frac{\partial}{\partial y}$ is not weighted Euler-like.

Weighted tubular neighborhoods

Any graded bundle $E \to N$ has a weighting, with $\nu_{\mathcal{W}}(E, N) = E$.

Definition

Given weighting along $N \subseteq M$, a weighted tubular neighborhood embedding is a map of pairs

$$\varphi\colon (\nu_{\mathcal{W}}(M, N), N) \to (M, N),$$

preserving filtrations, with $\nu_{\mathcal{W}}(\varphi) = \mathrm{id}$.

Theorem

Every (complete) weighted Euler-like vector field X gives unique tubular neighborhood embedding $\varphi\colon \nu_{\mathcal{W}}(M, N) \to M$ such that

$$\varphi^* X = \mathcal{E}.$$

Unweighted setting: Bursztyn-Lima-M (2019); see also Dang (2016).

Weighted tubular neighborhoods

Outline of proof: Following Bursztyn-Lima-M, Haj-Higson:

- $\widetilde{X}_{[0]} \in \mathfrak{X}(\mathcal{D}_{\mathcal{W}}(M,N))_{[0]}$ interpolates X at $t=1$ with $\mathcal{E}$ at $t=0$.
- Since X weighted Euler like,

$$\frac{\partial}{\partial t} + \frac{1}{t}\widetilde{X}_{[0]} \in \mathfrak{X}(M \times \mathbb{R}^{\times})$$

  extends smoothly to $W \in \mathfrak{X}(\mathcal{D}_{\mathcal{W}}(M,N))$.
- Since

$$\pi_* W = \frac{\partial}{\partial t}, \quad [W, \widetilde{X}_{[0]}] = 0,$$

  the flow of W takes fibers to fibers, and preserves $\widetilde{X}_{[0]}$.
- Take its time-1-flow to define

$$\varphi\colon \underbrace{\nu_{\mathcal{W}}(M,N)}_{\pi^{-1}(0)} \longrightarrow \underbrace{M}_{\pi^{-1}(1)}.$$

Weighted tubular neighborhoods

Example ($\mathbb{R}^2$ with weights $1, 2$)

$$X = (x + y^m)\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$$

is weighted Euler-like for all $m \in \mathbb{N}$. By theorem, there is a diffeomorphism φ such that

$$\varphi^* X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}.$$

Explictly, such a diffeomorphism is given by

$$\varphi(x, y) = \big(x + \frac{1}{1 - 2m}y^m,\ y\big).$$

Weighted tubular neighborhoods

Example (Isotropic embedding theorem)

Let (M, ω) be symplectic. For $N \subseteq M$ isotropic, get:

- Order $r = 2$ weighting, defined by $TN^\omega / TN \subseteq \nu(M, N)$
- $\omega \in \Omega^2(M)_{(2)} \;\rightsquigarrow\; \omega_{[2]} \in \Omega^2(\nu_{\mathcal{W}}(M, N))_{[2]}$.
- Pick $\alpha \in \Omega^1(M)_{(2)}\colon \mathrm{d}\alpha = \omega$
- Let $X \in \mathfrak{X}(M)_{(0)}\colon\; \iota_X \omega = 2\alpha$.
- X is Euler-like $\rightsquigarrow$ weighted tubular neighborhood embedding

$$\phi\colon \nu_{\mathcal{W}}(M, N) \to M$$

  such that $\phi^* \omega = \omega_{[2]}$.

This is a (small) improvement over *Weinstein's isotropic embedding theorem*.

5. Lie filtrations

Lie filtrations

Certain types of *hypo-elliptic* operators are defined using a *Lie filtration* on M

$$0 = H_{-1} \subseteq \cdots H_{-r} \subseteq TM;$$

i.e. $[\Gamma(H_{-i}), \Gamma(H_{-j})] \subseteq \Gamma(H_{-(i+j)})$. Then

$$\text{gr}\, TM = \bigoplus H_{-i}/H_{-i+1}$$

is a family of nilpotent Lie algebras, exponentiates to osculating tangent bundle $T_{\mathcal{W}}M \to M$.

Example

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$

define a Lie filtration

$$H_{-1} = \text{span}\{X\}, \quad H_{-2} = \text{span}\{X, Y\}, \quad H_{-3} = T\mathbb{R}^3.$$

The Kolmogorov operator $Y - X^2$ is hypo-elliptic.

Lie filtrations

Lie filtrations $0 = H_{-1} \subseteq \cdots H_{-r} \subseteq TM$ give weightings:

1. Lifts
$$X^{(-i)} \in \mathfrak{X}(T_r M), \quad X \in \Gamma(H_{-i})$$
span a nilpotent Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{X}(T_r M)$, $\rightsquigarrow$ graded subbundle
$$P = G \cdot M \subseteq T_r M.$$
Its fibers $P|_m$ define weightings for $(M, \{m\})$, and
$$T_{\mathcal{W}} M = \bigcup_{m \in M} \nu_{\mathcal{W}}(M, \{m\}) = P/\sim.$$
2. More generally, for 'nice' submanifolds $N \subseteq M$,
$$Q = P|_N \subseteq T_r M$$
defines a weighting, resulting in $\nu_{\mathcal{W}}(M, N)$ (see Haj-Higson).

Lie filtrations

Osculating tangent groupoid:

- Consider product Lie filtration on $M \times M$.
- Get *multiplicative* weighting along $\Delta_M \subseteq M \times M$.
- One finds $\nu_{\mathcal{W}}(M \times M, M_\Delta) = T_{\mathcal{W}} M$.
- So, get a weighted normal bundle

$$\mathcal{D}_{\mathcal{W}}(M \times M, M_\Delta) = T_{\mathcal{W}} M \sqcup (M \times M \times \mathbb{R}^\times).$$

See Choi-Ponge, van Erp-Yuncken, Haj-Higson, Mohsen.

Used by van Erp-Yuncken to obtain pseudo-differential calculus for Lie filtrations (along lines of Debord-Skandalis).

Current project (with Dan Hudson): Develop weighted conormal distributions for general weighted submanifolds $N \subseteq M$.

Further directions

Singular Lie filtrations

- Singular Lie filtrations:

$$\mathfrak{X}(M) = \mathcal{H}_{-r} \supseteq \mathcal{H}_{-r+1} \supseteq \cdots \supseteq \mathcal{H}_0, \qquad [\mathcal{H}_{-i}, \mathcal{H}_{-j}] \subseteq \mathcal{H}_{-(i+j)}$$

  (each $\mathcal{H}_{-i}$ locally finitely generated).

- Example: Martinez-Carnot structure
  $\mathcal{H}_{-1} = \mathcal{H},\ \mathcal{H}_{-2} = \mathcal{H} + [\mathcal{H}, \mathcal{H}],\ \mathcal{H}_{-3} = \mathfrak{X}(M)$ with

$$\mathcal{H} = \mathrm{span}\{\frac{\partial}{\partial x},\ \frac{\partial}{\partial y} + x^2 \frac{\partial}{\partial z}\}.$$

- Applications to hypo-elliptic operators: Androulidakis-van Erp-Mohsen-Yuncken.
- A singular Lie filtration gives singular foliation of $T_r M \rightsquigarrow$ many examples of weightings.

Further directions

Multi-weightings

Submanifolds N_1, N_2 intersect cleanly if $N_1 \cap N_2$ is submanifold with $T(N_1 \cap N_2) = TN_1 \cap TN_2$. In this case, can consider bi-weightings given by bi-filtrations

$$C^\infty(M)_{(i_1,i_2)}$$

(functions vanishing to weighted order i_a on N_a). This leads to

- Double normal bundle $\nu_{\mathcal{W}}(M, N_1, N_2)$,
- Double deformation spaces $\mathcal{D}_{\mathcal{W}}(M, N_1, N_2) \xrightarrow{\pi} \mathbb{R}^2$,
- Double tubular neighborhood embeddings
- Double tangent groupoid
- ....

More generally, multi-weightings along k-fold clean intersections.

Thanks!