The K-theory of quantized CW-complexes

(joint with: P. Hajac, T. Maszczyk, A. Sheu, B. Zieliński)

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19/01/2021

Global Noncommutative Geometry Seminar 2021

Introduction

- If a quantum space is obtained by "gluing" quantum spaces X and Y, there is a Mayer-Vietoris sequence relating its K-theory to that of X and Y.
- Examples are (quantized) CW complexes, obtained by iteratively gluing "cells". This idea allows to relate $K_\bullet(C(\mathbb{CP}^n_q))$ to $K_\bullet(C(\mathbb{CP}^{n-1}_q))$.
- Multipullback quantum projective spaces $\mathbb{CP}^n_{\mathcal{T}}$ don't have a CW complex structure. In:

  F. D'Andrea, P.M. Hajac, T. Maszczyk, A. Sheu & B. Zieliński, *Distinguished bases in the K-theory of $\mathbb{CP}^n_{\mathcal{T}}$*, arXiv:2002.09015 [math.KT].

  by using a tubular neighbourhood theorem we show how to construct generators of $K_\bullet(C(\mathbb{CP}^n_{\mathcal{T}}))$ to $K_\bullet(C(\mathbb{CP}^{n-1}_{\mathcal{T}}))$ and find a distinguished set of generators.
- In progress: generalization to spaces that are quantum CW complexes up to homotopy.
- By relating $\mathbb{CP}^n_q$ to $\mathbb{CP}^n_{\mathcal{T}}$ we find generators of $K_0(C(\mathbb{CP}^n_{\mathcal{T}}))$ represented by equivariant "vector bundles".

Gluing and pullbacks

A commutative diagram (of sets):

$$\begin{array}{ccc} P & \xleftarrow{\;j\;} & Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle f} \\ X & \xleftarrow[\;g\;]{} & Z \end{array}$$

is a pushout diagram if the map

$$P \xleftarrow{\;i\sqcup j\;} \frac{X \sqcup Y}{f(z) \sim g(z)} \quad \text{is iso.}$$

We get P by "gluing" X and Y along Z.

A commutative diagram of C^*-algebras:

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[\;g\;]{} & D \end{array}$$

is a pullback diagram if the map

$$A \xrightarrow{\;i\times j\;} \{(b,c) \in B \times C : f(b) = g(c)\} =: B \times_D C$$

is an isomorphism.

Applications? Get K-theory recursively $\Rightarrow$ Milnor's connecting homomorphism.

Mayer-Vietoris and pushouts

If $\{X, Y\}$ is an open cover of a smooth n-manifold P, one has the pushout diagram:

$$\begin{array}{ccc} P & \longleftarrow & Y \\ \uparrow & & \uparrow \\ X & \longleftarrow & Z := X \cap Y \end{array}$$

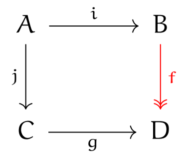
a short exact sequence of k-forms, and a long exact sequence in cohomology

$$0 \longrightarrow H^0_{dR}(P) \longrightarrow H^0_{dR}(X) \oplus H^0_{dR}(Y) \longrightarrow H^0_{dR}(Z) \longrightarrow$$
$$\longrightarrow H^1_{dR}(P) \longrightarrow H^1_{dR}(X) \oplus H^1_{dR}(Y) \longrightarrow H^1_{dR}(Z) \longrightarrow$$
$$\longrightarrow H^2_{dR}(P) \longrightarrow \;\ldots\ldots\ldots\ldots\ldots\ldots\; \longrightarrow H^n_{dR}(Z) \longrightarrow 0$$

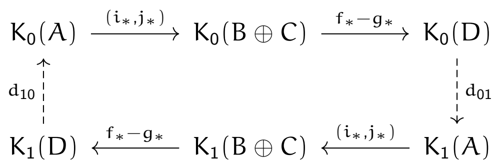
This holds for more general (co)homology theories (e.g. singular) and one-injective pushout diagrams (e.g. of CW complexes).

Mayer-Vietoris in K-theory

From a one-surjective pullback diagram of C^*-algebras



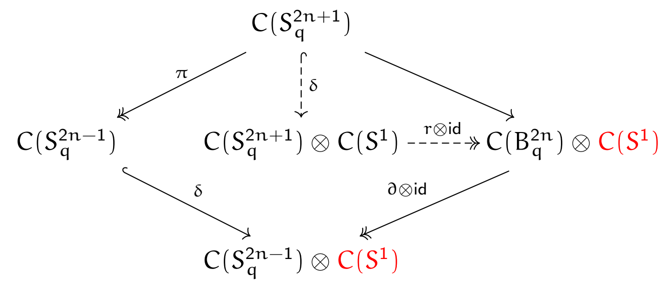
we get a six-term exact sequence



with d_{10}, d_{01} the "connecting homomorphisms".

Vaksman-Soibelman quantum spheres

There is a $U(1)$-equivariant pullback diagram:



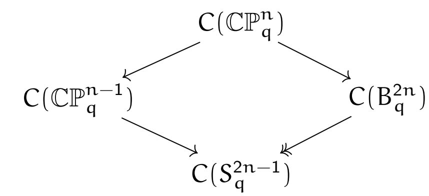
$\delta = C(S^1)$-coaction

$\partial =$ evaluation at the boundary

$r =$ restriction to half equator

$\pi =$ killing one complex "coordinate"

The $U(1)$-invariant part is automatically a (one-surjective) pullback diagram:



Holds more generally for trimmable graph C*-algebras. [Arici, D'Andrea, Hajac, Tobolski, 2018]

Multipullback quantum spaces

[Hajac, Nest, Pask, Sims, Zieliński, 2018]

- $C(S_H^{2n+1})$ is generated by commuting partial isometries $s_0, \ldots, s_n$ with relation

$$\prod_{i=0}^{n} (1 - s_i s_i^*) = 0$$

- $t :=$ right unilateral shift on $\ell^2(\mathbb{N})$, generator of $\mathcal{T} = C(B_q^2)$ ("quantum disk").
- $\mathcal{T}^{\otimes n+1}$ generated by

$$t_i := \underbrace{1 \otimes \ldots \otimes 1}_{i \text{ times}} \otimes t \otimes \underbrace{1 \otimes \ldots \otimes 1}_{n-i \text{ times}} \qquad \forall\, i = 0, \ldots, n.$$

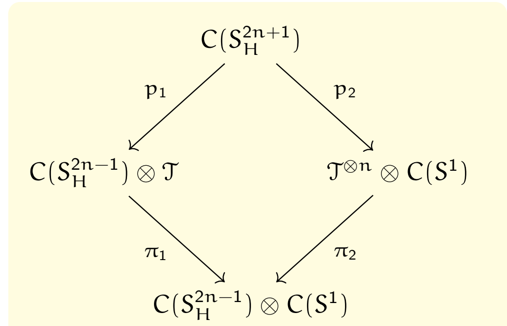
- There is an obvious $U(1)$-action on $C(S_H^{2n+1})$ and $\mathcal{T}^{\otimes n+1}$.
- $C(\mathbb{CP}_{\mathcal{T}}^n) := C(S_H^{2n+1})^{U(1)}$.
- There is a $U(1)$-equivariant short exact sequence

$$0 \to \mathcal{K}(\ell^2(\mathbb{N}^{\otimes n+1})) \to \mathcal{T}^{\otimes n+1} \xrightarrow{\sigma_n} C(S_H^{2n+1}) \to 0$$

where $\boxed{\sigma_n(t_i) := s_i}$ for all $i = 0, \ldots, n$.

Lemma [Hajac, Nest, Pask, Sims, Zieliński, 2018]

For every $n \geqslant 1$ the diagram



$$p_1(s_i) := \begin{cases} s_i \otimes 1 & \forall\, 0 \leqslant i < n \\ 1 \otimes t & \text{if } i = n \end{cases}$$

$$p_2(s_i) := \begin{cases} t_i \otimes 1 & \forall\, 0 \leqslant i < n \\ 1 \otimes u & \text{if } i = n \end{cases}$$

$$\pi_1 := id \otimes \sigma_0$$

$$\pi_2 := \sigma_{n-1} \otimes id$$

is a $U(1)$-equivariant pullback diagram w.r.t. the diagonal $U(1)$-action on each vertex.

We can tensor everywhere by $\mathcal{T}^{\otimes k} =: \mathcal{T}^k$ and get a new pullback diagram:

$$C(S_H^{2n+1})_\bullet \otimes \mathcal{T}^k_\bullet \xrightarrow{p_1 \otimes \mathrm{id}_{\mathcal{T}^k}} (C(S_H^{2n-1})_\bullet \otimes \mathcal{T}_\bullet) \otimes \mathcal{T}^k_\bullet$$

$$C(S_H^{2n+1})_\bullet \otimes \mathcal{T}^k_\bullet \xrightarrow{p_2 \otimes \mathrm{id}_{\mathcal{T}^k}} \mathcal{T}^{\otimes n}_\bullet \otimes C(S^1)_\bullet \otimes \mathcal{T}^k_\bullet$$

$$(C(S_H^{2n-1})_\bullet \otimes \mathcal{T}_\bullet) \otimes \mathcal{T}^k_\bullet \xrightarrow{\pi_1 \otimes \mathrm{id}_{\mathcal{T}^k}} C(S_H^{2n-1})_\bullet \otimes C(S^1)_\bullet \otimes \mathcal{T}^k_\bullet$$

$$\mathcal{T}^{\otimes n}_\bullet \otimes C(S^1)_\bullet \otimes \mathcal{T}^k_\bullet \xrightarrow{\pi_2 \otimes \mathrm{id}_{\mathcal{T}^k}} C(S_H^{2n-1})_\bullet \otimes C(S^1)_\bullet \otimes \mathcal{T}^k_\bullet$$

…then gauge the $U(1)$-action using the map:

$$\phi : A_\bullet \otimes C(S^1)_\bullet \otimes B_\bullet \to A \otimes C(S^1)_\bullet \otimes B$$
$$a \otimes f \otimes b \mapsto a_{(0)} \otimes a_{(1)} f b_{(-1)} \otimes b_{(0)}$$

and get the $U(1)$-pullback diagram:

$$C(S_H^{2n+1})_\bullet \otimes \mathcal{T}^k_\bullet \xrightarrow{p_1 \otimes \mathrm{id}_{\mathcal{T}^k}} C(S_H^{2n-1})_\bullet \otimes \mathcal{T}^{k+1}_\bullet$$

$$C(S_H^{2n+1})_\bullet \otimes \mathcal{T}^k_\bullet \xrightarrow{\phi \circ (p_2 \otimes \mathrm{id}_{\mathcal{T}^k})} \mathcal{T}^n \otimes C(S^1)_\bullet \otimes \mathcal{T}^k$$

$$C(S_H^{2n-1})_\bullet \otimes \mathcal{T}^{k+1}_\bullet \xrightarrow{\phi \circ (\pi_1 \otimes \mathrm{id}_{\mathcal{T}^k})} C(S_H^{2n-1}) \otimes C(S^1)_\bullet \otimes \mathcal{T}^k$$

$$\mathcal{T}^n \otimes C(S^1)_\bullet \otimes \mathcal{T}^k \xrightarrow{\phi \circ (\pi_2 \otimes \mathrm{id}_{\mathcal{T}^k}) \circ \phi^{-1} = \pi_2 \otimes \mathrm{id}_{\mathcal{T}^k}} C(S_H^{2n-1}) \otimes C(S^1)_\bullet \otimes \mathcal{T}^k$$

For $k = 0$ and for all $n \geqslant 1$, the $U(1)$-invariant part gives the pullback diagram:

$$C(\mathbb{CP}^n_{\mathcal{T}}) \xrightarrow{p_1} (C(S_H^{2n-1}) \otimes \mathcal{T})^{U(1)}$$

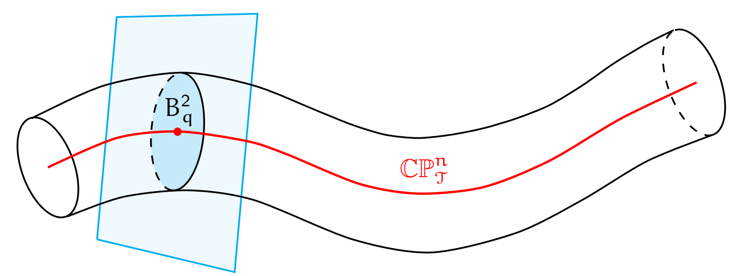
$$C(\mathbb{CP}^n_{\mathcal{T}}) \xrightarrow{\phi \circ p_2} \mathcal{T}^{\otimes n}$$

$$(C(S_H^{2n-1}) \otimes \mathcal{T})^{U(1)} \xrightarrow{\phi \circ (\mathrm{id} \otimes \sigma_0)} C(S_H^{2n-1})$$

$$\mathcal{T}^{\otimes n} \xrightarrow{\sigma_{n-1}} C(S_H^{2n-1})$$

Observe: no cell complex structure (compare with Vaksman-Soibelman).

A tubular neighbourhood theorem



Theorem

For all $n \geqslant 0$, the map

$$C(\mathbb{CP}^n_{\mathcal{T}}) \xrightarrow{\mathrm{id} \otimes 1_{\mathcal{T}}} C\big(TN(\mathbb{CP}^n_{\mathcal{T}})\big) := \big(C(S_H^{2n+1}) \otimes \mathcal{T}\big)^{U(1)}$$

is a weak equivalence.

Terminology

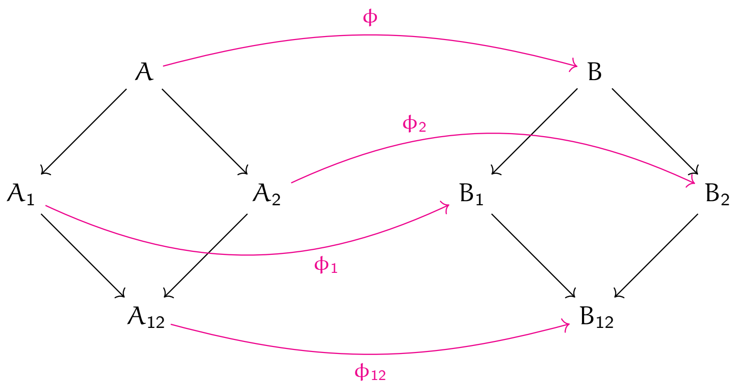
A $*$-hom $\phi : A \to B$ is called a weak equivalence if $\phi_* : K_\bullet(A) \to K_\bullet(B)$ is iso.

Observation

A, B unital with $K_0 = \mathbb{Z}[1]$ and $K_1 = 0 \implies$ any unital $*$-hom $A \to B$ is a weak equivalence.

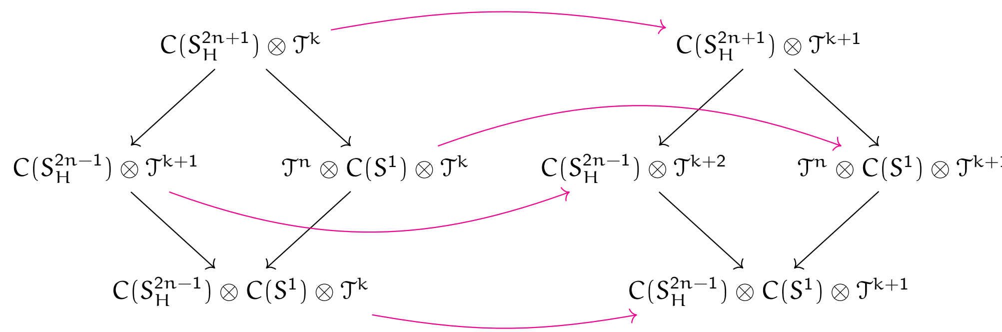
Comparison Lemma [Farsi, Hajac, Maszczyk, Zieliński, 2017]

In a commutative diagram



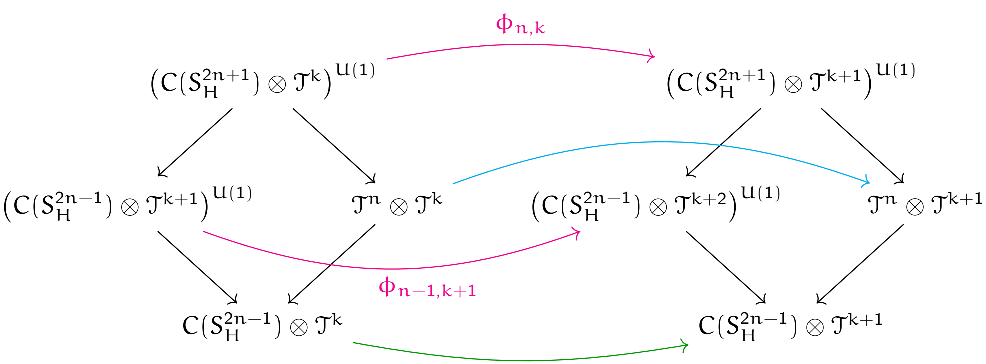
if the squares are pullback diagrams and $\phi_1, \phi_2, \phi_{12}$ are weak equivalences, then so is ϕ.

Proof of the Theorem. Consider the commutative diagram:



The squares are pullbacks, the $U(1)$-action is diagonal on top-left vertices and central on bottom-right, horizontal arrows are $\mathrm{id} \otimes 1_{\mathcal{T}}$.

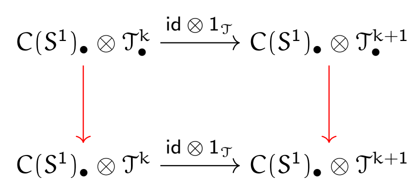
The $U(1)$-invariant part gives:



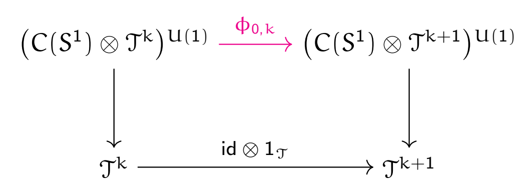
- Blue line = weak equivalence by the above-mentioned Observation.
- Green line = tensor product of weak equivalences ($\mathrm{id}_{C(S^{2n-1}_H)}$ and $\mathrm{id}_{\mathcal{T}^k} \otimes 1_{\mathcal{T}}$).
- If we prove that $\phi_{0,k}$ is a weak equivalence for all $k \geqslant 0$, the Comparison Lemma + induction on n imply that $\phi_{n,k}$ is a weak equivalence for all $n, k \geqslant 0$.
- In particular $\phi_{n,0} = \mathrm{id} \otimes 1_{\mathcal{T}} : C(\mathbb{CP}^n_{\mathcal{T}}) \to C\big(TN(\mathbb{CP}^n_{\mathcal{T}})\big)$ is a weak equivalence.

It remains to prove that $\phi_{0,k}$ is a weak equivalence.

$U(1)$-equiv. commutative diagram:



$U(1)$-invariant part:

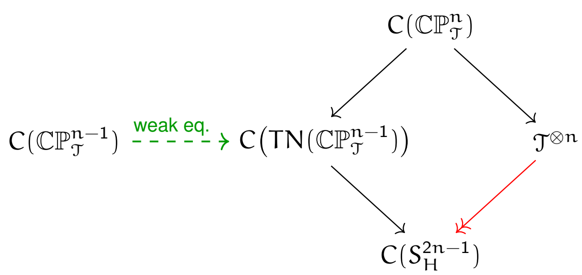


(red arrows: $a \otimes b \mapsto ab_{(-1)} \otimes b_{(0)}$)

Since the vertical arrows are isomorphisms and (by the above Observation) the bottom arrows are weak equivalences, $\phi_{0,k}$ is a weak equivalence. ∎

Milnor and K-groups

Using the six-term exact sequence associated to the one-surjective pullback diagram



and the tubular neighbourhood theorem one proves that:

Proposition

For all $n \geqslant 1$:

$$K_0(C(\mathbb{CP}^n_{\mathcal{T}})) \simeq K_0(C(\mathbb{CP}^{n-1}_{\mathcal{T}})) \oplus d_{10}\big(K_1(C(S^{2n-1}_H))\big)$$

where d_{10} is Milnor's connecting homomorphism.

A comparison theorem

For all $n \geqslant 0$ there is a $U(1)$-equivariant weak equivalence:

$$C(S^{2n+1}_q) \to C(S^{2n+1}_H)$$

Its restriction and corestriction to fixed point algebras:

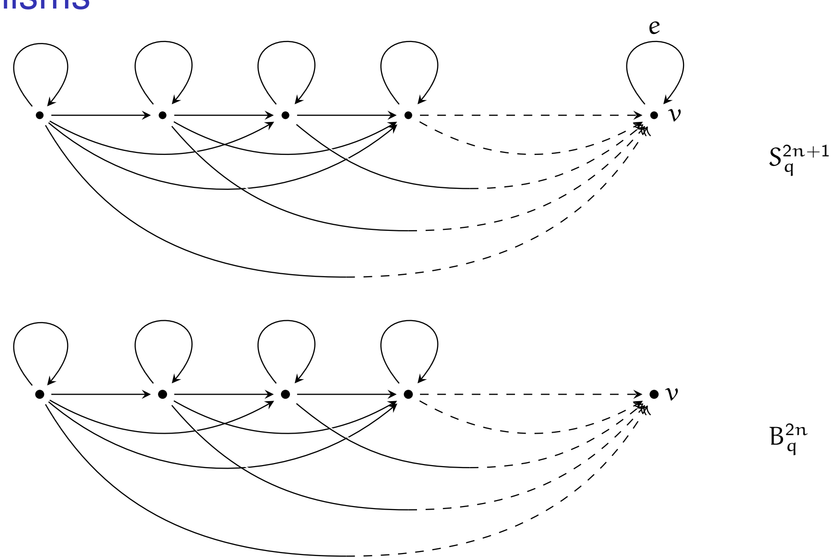
$$C(\mathbb{CP}^n_q) \to C(\mathbb{CP}^n_{\mathcal{T}})$$

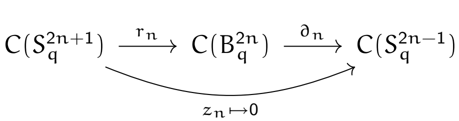
is a weak equivalence as well.

It maps vector bundles associated to $S^{2n+1}_q \xrightarrow{U(1)} \mathbb{CP}^n_q$, generators of $K_0(C(\mathbb{CP}^n_q))$, to v. bundles associated to $S^{2n+1}_H \xrightarrow{U(1)} \mathbb{CP}^n_{\mathcal{T}}$,* proving that the latter generate $K_0(C(\mathbb{CP}^n_{\mathcal{T}}))$.

*Pushforward commutes with association, cf. [Hajac, Maszczyk, 2018].

Morphisms

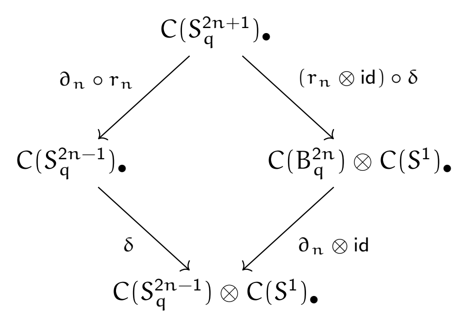




$r_n : S_e \mapsto P_v$

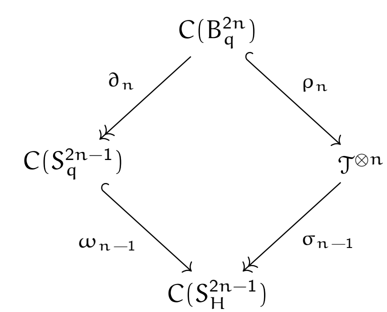
$\partial_n : t^{-1}(v) \mapsto 0, P_v \mapsto 0$

A $U(1)$-equivariant pullback diagram:



[Arici, D'Andrea, Hajac, Tobolski, 2018]

Spherical vs. non-spherical balls:



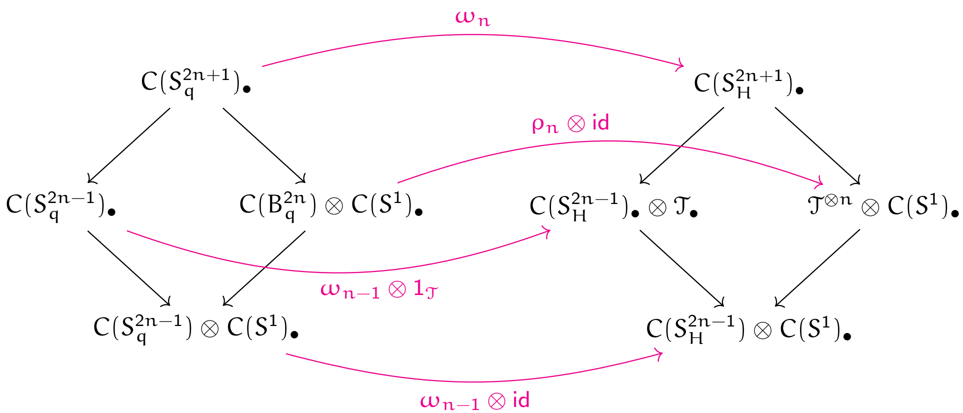
On generators:

$$\rho_n(S_{e_{ij}}) := t_i t_j t_j^* \prod_{k=0}^{j-1}(1 - t_k t_k^*) \qquad \forall\, 0 \leqslant i \leqslant j < n,$$

$$\rho_n(S_{e_{in}}) := t_i \prod_{k=0}^{n-1}(1 - t_k t_k^*) \qquad \forall\, 0 \leqslant i < n,$$

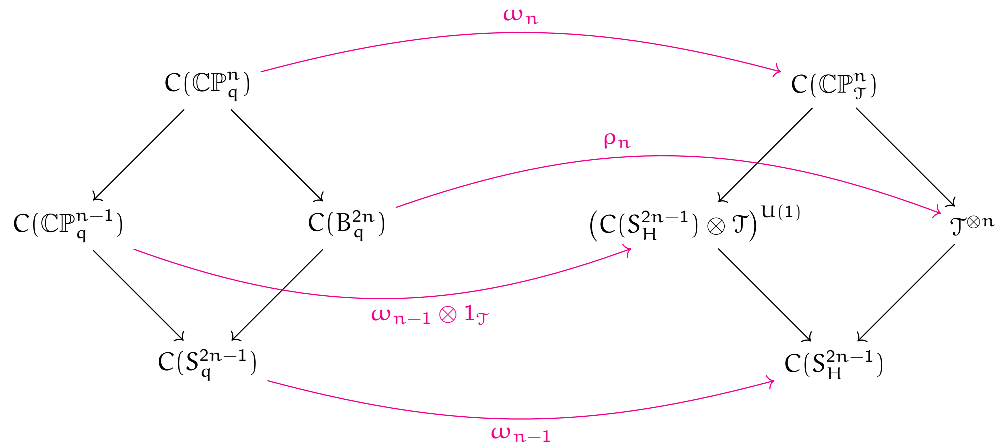
$$\omega_{n-1}(S_{e_{ij}}) := s_i s_j s_j^* \prod_{k=0}^{j-1}(1 - s_k s_k^*) \qquad \forall\, 0 \leqslant i \leqslant j < n.$$

Proof of the Theorem.
With an explicit check on generators one proves the commutativity of the diagram:



The left and right square are pullbacks. Comparison Lemma + induction on n prove that ω_n is a weak equivalence (ρ_n is a weak equivalence due to the Observation, tensor product of weak equivalences is a weak equivalence by functoriality of to Kunneth formula).

Passing to $U(1)$-fixed point algebras one gets we get the commutative diagram:



Using again Comparison Lemma + induction on n we prove that the top horizontal map is a weak equivalence. ■

Noncommutative Geometry and Physics

Monthly talks:

`https://researchseminars.org/seminar/NCGandPH`

and a special issue of J Phys A.

Organizers: Paolo Aschieri, Edwin Beggs, FD, Emil Prodan, Andrzej Sitarz.

Upcoming talks (Monday at 16:00):

Jan 25	Shahn Majid	Quantum groups and quantum spacetime models at the Planck scale
Feb 22	Stefan Waldmann	TBA
Mar 29	Emil Prodan	TBA
Apr 26	Richard Szabo	TBA