# Kaplansky's conjectures

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# Promo

Upcoming talk on the computational side of this story at the Sydney Mathematical Research Institute on “5” October (registration needed).

These slides are available at www.gilesgardam.com/slides/gncg.pdf.

# Kadison–Kaplansky conjecture

## Kadison–Kaplansky idempotent conjecture

Let $G$ be a torsion-free discrete group. Then the reduced group $C^*$-algebra of $G$ has no idempotents other than 0 and 1.

Equivalently, the spectrum of every element is connected.

Torsion-free is needed: if $g^n = 1$ then $\frac{1}{n}(1 + g + \cdots + g^{n-1})$ is an idempotent.

Pimsner–Voiculescu proved the conjecture for free groups (1980). It was subsequently realized that it follows in general from the Baum–Connes conjecture.

Even for $\mathbb{C}[G]$ the conjecture is wide open, and likewise for the group algebra (a.k.a. group ring) $K[G]$ over an arbitrary field $K$.

# Kaplansky conjectures

We can strengthen the idempotent conjecture with:

**Kaplansky zero divisor and unit conjectures (Higman 1940)**

Let $G$ be a torsion-free group and let $K$ be a field. Then the group ring $K[G]$

- has no (non-zero) zero divisors, i.e., $ab = 0 \implies a = 0$ or $b = 0$, and
- has only trivial units, i.e., $ab = ba = 1 \implies a = kg$ for $k \in K \setminus \{0\}$ and $g \in G$.

(For group $C^*$-algebras, these are false by functional calculus.)

The unit conjecture is false (G. 2021).

# A note on characteristic

Sometimes one can go back and forth between positive characteristic and zero characteristic.

For a non-principal ultrafilter $\omega$ we have

$$\mathbb{C} \cong \prod_{\omega} \overline{\mathbb{F}}_p.$$

If we prove a Kaplansky conjecture for all finite fields, then we have it for $\overline{\mathbb{F}}_p = \cup_k \mathbb{F}_{p^k}$ and thus $\mathbb{C}$. If we disprove a Kaplansky conjecture for all finite fields *with uniformly bounded support of the counterexamples*, then we have disproved it for $\mathbb{C}$.

# Gromov's dichotomy

When we are dealing with statements for all groups (or all countable groups, or all torsion-free groups, or...) we must not forget:

**Gromov's dichotomy**

Any statement for all countable groups is either trivial or false.

However, we have to take it with a grain of salt as well:

- The trace of a non-trivial idempotent in $\mathbb{C}[G]$ lies in $(0, 1)$ (Kaplansky 1965) and is rational (Zalesskii 1972).
- The group ring of a torsion-free group over a field $K$ is prime: if $arb = 0$ for all $r \in K[G]$ then $a = 0$ or $b = 0$ (Connell 1963).

Kaplansky's theorem implies that $\mathbb{C}[G]$ is *von Neumann finite* (a.k.a. *directly finite*): $ab = 1 \implies ba = 1$ since otherwise $ba$ is a non-trivial idempotent of trace 1. In positive characteristic this is open but known for sofic groups (Elek–Szabo 2004).

# Relationship between the conjectures

For each individual group ring $K[G]$ we have

unit conjecture $\implies$ zero divisor conjecture $\implies$ idempotent conjecture

A non-trivial idempotent $x$ is a zero divisor since $x(x-1) = x^2 - x = 0$.

Turning a zero divisor into a non-trivial unit is a little more work. Suppose that $ab = 0$ for some non-zero $a, b \in K[G]$. Since $K[G]$ is prime we can find $c \in K[G]$ such that $bca \neq 0$. Now $(bca)^2 = bc(ab)ca = 0$ so that $(1 + bca)(1 - bca) = 1$ and we have non-trivial units (after quickly thinking about the minor technicality in characteristic 2).

# Unit conjecture counterexample

**Theorem (G. 2021)**

*Let $P$ be the torsion-free virtually abelian group defined by the presentation $\langle\, a, b \mid b^{-1}a^2b = a^{-2},\ a^{-1}b^2a = b^{-2}\,\rangle$ and set $x = a^2, y = b^2, z = (ab)^2$. Set*

$$p = (1+x)(1+y)(1+z^{-1}), \qquad q = x^{-1}y^{-1} + x + y^{-1}z + z,$$
$$r = 1 + x + y^{-1}z + xyz, \qquad s = 1 + (x + x^{-1} + y + y^{-1})z^{-1}.$$

*Then $p + qa + rb + sab$ is a non-trivial unit in the group ring $\mathbb{F}_2[P]$.*

Its inverse is $x^{-1}p^a + x^{-1}qa + y^{-1}rb + z^{-1}s^a ab$. The computation is $\sim 3$ pages.

Alan Murray has generalized the construction to cover every positive characteristic arXiv:2106.02147 but characteristic zero remains open.

# Unit conjecture counterexample

$P$ is the Hantzsche–Wendt crystallographic group. The subgroup $\langle x, y, z \rangle \cong \mathbb{Z}^3$ is normal with quotient $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Picking a suitable isometric (left) action on $\mathbb{R}^3$ realizes the polynomials $p = (1+x)(1+y)(1+z^{-1})$, $q = x^{-1}y^{-1} + x + y^{-1}z + z$, $r = 1 + x + y^{-1}z + xyz$ and $s = 1 + (x + x^{-1} + y + y^{-1})z^{-1}$ as follows

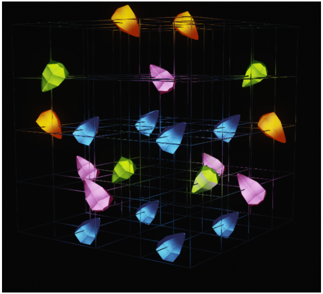

Figure 1: Source: quantamagazine.org

# The group of units

The unit conjecture predicts that $(K[G])^\times \cong K^\times \times G$.

**Corollary (G. 2021)**

*The group of units of $\mathbb{F}_2[P]$ is a torsion-free linear group that is not finitely generated and contains non-abelian free subgroups.*

For infinite generation we have to first turn the counterexample into a family of units, which we can do via affine rescaling. We then map to $\mathbb{F}_2[D_\infty]$ and use Mirowicz's calculation of

$$(\mathbb{F}_2[D_\infty])^\times \cong (*_{j\in\mathbb{Z}} \oplus_{i\in\mathbb{N}^+} \mathbb{Z}/2) \rtimes D_\infty.$$

# Mildness of $P$

The group $P$ is extraordinarily mild (disregarding the non-trivial units it provides) as it is virtually abelian.

It satisfies the zero divisor conjecture (Brown 1976, Cliff 1980).

Since it satisfies the Farrell–Jones conjecture, the Whitehead group $\mathrm{Wh}^K(P)$ is trivial. This means that all units are "stably trivial": we can trivialize them in $K_1(K[P]) = \mathrm{GL}_\infty(K[P])^{ab}$.

Why would one try to disprove the unit conjecture with $P$? There is a naive property that implies the conjecture that only a small handful of groups – including $P$ – are known not to have.

# What is known

Kaplansky's conjectures follow from other bold conjectures for torsion-free groups that are subject to Gromov scepticism (or were for a few decades).

- Unique products $\implies$ unit conjecture
- Atiyah conjecture $\implies$ zero divisor conjecture over $\mathbb{C}$
- Baum–Connes conjecture or Farrell–Jones conjecture $\implies$ idempotent conjecture over $\mathbb{C}$

A group $G$ *has unique products* if for finite subsets $A, B \subset G$ there is always some element uniquely expressible as $ab$ for $a \in A, b \in B$.

The zero divisor conjecture actually holds for elementary amenable groups over any field.

# What is known (continued)

**Lemma (Botto Mura–Rhemtulla 1975)**

Left-orderability implies unique products.

A group is *left-orderable* if it admits a total order invariant under left multiplication. Left-orderable groups include free groups, torsion-free nilpotent groups, torsion-free one-relator groups, Thompson's group $F$...

**Proof.**

Enumerate the finite subsets $A, B \subset G$ as $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_n$. Then $a_i b_1 < a_i b_j$ for all $j \neq 1$ and thus the minimal product $a_i b_j$ must have $j = 1$. The $m$ possibilities $a_1 b_1, \ldots a_m b_1$ are distinct group elements, so the minimal product is unique. (NB: it will *not* be $a_1 b_1$ in general, because we only assume the order to be left-invariant!) □

# Non-unique product groups

The concept of unique product groups appears in Higman's thesis and was introduced by Rudin and Schneider in 1964.

The first example of a torsion-free group without unique products was constructed by Rips and Segev in 1987 using small cancellation. Shortly thereafter, Promislow showed that the group $P$ of the unit conjecture counterexample contains a 14-element set $A$ such that $A \cdot A$ has no unique product.

# Non-unique product groups (continued)

The known groups without unique products come in two flavours:

- small cancellation: Rips–Segev (1987), Steenbock (2015), Gruber–Martin–Steenbock (2015), Arzhantseva–Steenbock (2014+)
- small presentation: Promislow (1988), Carter (2014), Soelberg (2018)

The examples of the second flavour are known to satisfy the zero divisor conjecture.

# What's next

An explicit counterexample to the zero divisor conjecture would need a new non-unique product group.

**Theorem (G. 2021)**

*The torsion-free group* $\langle\, a, b \,|\, aba^2b^{-1}a^2b^{-2}, ab^3ab^4a^{-1}b \,\rangle$ *does not have the unique product property. It is an* $\widetilde{A}_2$ *lattice and thus has property (T).*

# Questions

- Can we prove the unit conjecture for *any* torsion-free group that does not have the unique product property?
- What about non-trivial units in $\mathbb{C}[P]$? Or $\mathbb{Z}[P]$?

# Questions?