Dauns-Hofmann-Kumjian-Renault Duality for Fell Bundles and Structured C*-Algebras

Tristan Bice

Institute of Mathematics of the Czech Academy of Sciences

Global Noncommutative Geometry Seminar (Asia-Pacific)
27th of September 2021

Brief History

- In the beginning, there was Gelfand (1941-43):

Commutative C*-Algebras	$\leftrightarrow$	Locally Compact Hausdorff Spaces
C*-Algebra Homomorphisms	$\leftrightarrow$	(Partial) Proper Continuous Maps

- What about non-commutative C*-algebras? Replace spaces with bundles a la Dauns-Hofmann (1969) and Varela (1974):

C*-Algebras (A, Z) with central Z	$\leftrightarrow$	C*-Bundles over LCH Spaces
Z-Preserving C*-Homomorphisms	$\leftrightarrow$	Continuous Base & Fibre Maps

- Problem: If the centre is trivial then so is the bundle.
- Alternative: Replace C*-bundles with saturated Fell line bundles a la Kumjian (1986) and Renault (2008):

C*-Algebras (A, C) with Cartan C $\leftrightarrow$ Fell Bundles over LCH Étale Groupoids

- Main Questions:
  - Can we also make the Kumjian-Renault duality functorial?
  - Can we even unify it with the Dauns-Hofmann-Varela duality?

Classic Gelfand Duality

- Motivating Question: How can we recover X from $\mathcal{C}_0(X)$?
- Each $x \in X$ defines a maximal closed ideal

$$I_x = \{a \in \mathcal{C}_0(x) : a(x) = 0\}.$$

  Equivalently, $a \in I_x$ iff $x \notin \mathrm{supp}(a) = \{x \in X : a(x) \neq 0\}$.
- This covers every $I \in \mathcal{I}(A)$(= maximal closed ideals of A).
- $x \mapsto I_x$ is a homeomorphism (hull-kernel topology on $\mathcal{I}(A)$).
- Answer: Maximal closed ideals of $\mathcal{C}_0(X)$ recover X.

- Abstract Extension: Start with a commutative C*-algebra A.
- Maximal closed ideals $\mathcal{I}(A)$ have LCH hull-kernel topology.
- Every $a \in A$ defines $\widehat{a} \in \mathcal{C}_0(\mathcal{I}(A))$ by

$$\widehat{a}(I) = a + I \in A/I \approx \mathbb{C}.$$

- Moreover, $a \mapsto \widehat{a}$ is a C*-algebra isomorphism.
- Summary: LCH spaces are ‘dual’ to commutative C*-algebras.

Milgram Approach

- On any semigroup A, define a transitive domination relation

$$a < b \qquad \Leftrightarrow \qquad \exists s \in A\ (a = asb).$$

- Take $A = \mathcal{C}_0(X)$ and note that

$$a < b \qquad \Leftrightarrow \qquad \mathrm{supp}(a) \Subset \mathrm{supp}(b)$$

($O \Subset N$ means $\exists$ compact K s.t. $O \subseteq K \subseteq N$).

- Each $x \in X$ defines an $<$-ultrafilter(=max $\downarrow$-directed $\uparrow$-set)

$$U_x = A \setminus I_x = \{a \in A : a(x) \neq 0\} = \{a \in A : x \in \mathrm{supp}(a)\}.$$

- This covers every $U \in \mathcal{U}(A)$(= ultrafilters in A).
- $x \mapsto U_x$ is a homeomorphism w.r.t. topology with basis

$$\mathcal{U}(a) = \{U \in \mathcal{U}(A) : a \in U\}.$$

Theorem ($\approx$ Milgram 1949)

X can be recovered from just the product structure of $\mathcal{C}_0(X)$.

Functoriality

- $\pi : X \to Y$ is proper if preimages of compacta are compact.
- If π is also continuous it induces $\widehat{\pi} : \mathcal{C}_0(Y) \to \mathcal{C}_0(X)$ by

$$\widehat{\pi}(f) = f \circ \pi.$$

- If X and Y are compact then $\widehat{\pi}(1_Y) = 1_X$.
- In general, $\widehat{\pi}$ is an approximately unital *-homomorphism.

Theorem

Every approximately unital *-homo is of this form for unique π.

- To capture general *-homomorphisms, one must consider continuous proper $\pi : O \to Y$ defined on open $O \subseteq X$. Then

$$\widehat{\pi}(f)(x) = \begin{cases} f(\pi(x)) & \text{if } x \in O \\ 0 & \text{if } x \in X \setminus O. \end{cases}$$

- In this way **CommC*** and **LCH** are dual categories.

Matrices

Theorem ($\approx$ Wedderburn 1908)

Every finite dimensional C*-algebra is a direct sum of matrices.

- Typically, *-homos between finite dimensional C*-algebras are defined by copying smaller matrices into larger ones, e.g.

$$\begin{bmatrix} a_{(1,1)} & a_{(1,2)} \\ a_{(2,1)} & a_{(2,2)} \end{bmatrix} \mapsto \begin{bmatrix} a_{(1,1)} & a_{(1,2)} & 0 & 0 \\ a_{(2,1)} & a_{(2,2)} & 0 & 0 \\ 0 & 0 & a_{(1,1)} & a_{(1,2)} \\ 0 & 0 & a_{(2,1)} & a_{(2,2)} \end{bmatrix}$$

- Viewing M_2 and M_4 as functions on $\{1,2\}^2$ and $\{1,2,3,4\}^2$, this also arises from a partial map π in the opposite direction

$$\pi(m,n) = \begin{cases} (m,n) & \text{if } m,n \leq 2 \\ (m-2,n-2) & \text{if } m,n \geq 3 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Star-Bijective Functors

- Note $G_n = \{1, \ldots, n\}^2$ is a groupoid where

$$(j,k)(k,l) = (j,l) \qquad \text{and} \qquad (j,k)^{-1} = (k,j).$$

- $\pi : G(\subseteq_{\text{subgrpd}} G_m) \to G_n$ is a functor if we have a function $p : M(\subseteq \{1, \ldots, m\}) \to \{1, \ldots, n\}$ such that

$$\pi(j,k) = (p(j), p(k)).$$

- π is star-bijective if the j^{th} star(=row or column) in G is mapped bijectively onto the $p(j)^{\text{th}}$ star in G_n, i.e.

$$S_j = \{(j,k) : (j,k) \in G\} \quad \overset{\pi}{\longleftrightarrow} \quad S_{p(j)} = \{(p(j), l) : l \leq n\}.$$

- E.g. previous $\pi : G(\subseteq G_4) \to G_2$ is a star-bijective functor.
- Note: Star-bijective functors are special kinds of (partial) ‘Zakrzewski morphisms’.

Monomial Preservation

- Star-bijective functors yield *-homos preserving monomials – matrices with ≤ 1 non-zero entry in each row and column, e.g.

$$\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \mapsto \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{bmatrix}$$

- Off-diagonal $\mathbb{T}$-multiple *-homos also preserve monomials, e.g.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \text{ or } \begin{bmatrix} a & ib \\ -ic & d \end{bmatrix}$$

Theorem

Every monomial-preserving *-homomorphism from M_m to M_n comes from a star-bijective functor and an off-diagonal $\mathbb{T}$-multiple.

Diagonal Preservation

Corollary

Every diagonal-preserving *-automorphism on M_n comes from a groupoid isomorphism and an off-diagonal $\mathbb{T}$-multiple.

- Diagonal-preserving does not suffice for general *-homos, e.g.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 & \frac{1}{\sqrt{2}}b & \frac{1}{\sqrt{2}}b \\ 0 & a & \frac{1}{\sqrt{2}}b & -\frac{1}{\sqrt{2}}b \\ \frac{1}{\sqrt{2}}c & \frac{1}{\sqrt{2}}c & d & 0 \\ \frac{1}{\sqrt{2}}c & -\frac{1}{\sqrt{2}}c & 0 & d \end{bmatrix}$$

preserves the diagonal (and off-diagonal) but not monomials:

$$\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \mapsto \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix}$$

Fell Bundles

- Problem: Find a general framework for $\mathcal{C}_0(X)$ and M_n.

Option 1: Consider functions with non-commutative ranges, i.e. $\mathcal{C}_0(\rho)$ for C*-bundle $\rho : B \twoheadrightarrow X$ (Dauns-Hofmann 1969).

Option 2: Consider functions with non-commutative domains, i.e. $\mathcal{C}_\mathrm{r}(G)$ for LCH étale groupoid G (Renault 1980).

Option 3: Do both, i.e. consider continuous sections $\mathcal{C}_\mathrm{r}(\rho)$ of Fell bundle $\rho : B \twoheadrightarrow G$ (Kumjian 1998).

Definition

A Fell bundle is a Banach bundle $\rho : B \twoheadrightarrow G$ such that

1. B is a topological *-semigroupoid with bilinear $\cdot$ and antilinear *.
2. G is an LCH étale groupoid and ρ is a *-isocofibration (meaning $\rho(ab) = \rho(a)\rho(b)$ whenever ab OR $\rho(a)\rho(b)$ is defined).
3. $\|ab\| \leq \|a\|\|b\|$, $\|b^*b\| = \|b\|^2$ and $b^*b \geq 0$, for all $a, b \in B$.

- $\rho^{-1}[G^0]$ is a C*-bundle, i.e. unit fibres are C*-algebras.
- ρ is a line bundle if all fibres are 1-dimensional.
- ρ is strongly saturated if all fibres contain invertibles.

Reduced C*-Algebras

- Let $\rho : B \twoheadrightarrow G$ be a Fell bundle.
- Given sections $a, b : G \to B$, define another section ab by

$$ab(g) = \sum_{g=jk} a(j)b(k),$$

  whenever the finite partial sums converge.
- Note ab is defined whenever a or b has compact support.
- Define $\|a\|_\infty = \sup_{g\in G} |a(g)|$. Also let $a^*(g) = a(g^{-1})^*$ and

$$\|a\|_2 = \sqrt{\|a^*a\|_\infty} \quad (= \infty \text{ if } a^*a \text{ is undefined})$$

$$\|a\|_{\mathsf{b}} = \sup_{\substack{\|f\|_2 \leq 1 \\ |\mathrm{supp}(f)|<\infty}} \|af\|_2 \quad (= \text{‘left-regular norm’})$$

- Then $\mathcal{C}_{\mathsf{b}}(\rho) = \{a \in \mathcal{C}(\rho) : \|a\|_{\mathsf{b}} < \infty\}$ is a Banach space (where $\mathcal{C}(\rho)$ denotes the continuous sections of ρ).
- The reduced C*-algebra of ρ is then defined by

$$\mathcal{C}_{\mathsf{r}}(\rho) = \mathrm{cl}_{\mathsf{b}}(\mathcal{C}_{\mathsf{c}}(\rho))$$

  (where $\mathcal{C}_{\mathsf{c}}(\rho) = \{a \in \mathcal{C}(\rho) : \mathrm{cl}(\mathrm{supp}(a)) \text{ is compact}\}$).

Groupoid Recovery: The ‘Right’ Question

- Let $\rho : B \twoheadrightarrow G$ be a (strongly) saturated Fell line bundle.
- Naïve Question: Can we recover G from $\mathcal{C}_r(\rho)$?
- No, can have $\mathcal{C}_r(\rho) \approx \mathcal{C}_r(\rho')$ even when $G \not\approx G'$.
- But in this case often $G^0 \not\approx G'^0$ as well. By Gelfand, this implies that we get different diagonal subalgebras

$$\mathcal{D}_r(\rho) = \{a \in \mathcal{C}_r(\rho) : \mathrm{supp}(a) \subseteq G^0\}.$$

- Better Question: Can we recover G from $(\mathcal{C}_r(\rho), \mathcal{D}_r(\rho))$?
- Yes if G is principal (Kumjian 1986)/effective (Renault 2008).
- What about other G? Problem: $\mathbb{Z}_2 \times \mathbb{Z}_2 \not\approx \mathbb{Z}_4$ even though $\mathcal{C}_r(\mathbb{Z}_2 \times \mathbb{Z}_2) \approx \mathcal{C}_r(\mathbb{Z}_4) \approx \mathbb{C}^4$ and $\mathcal{C}_0(\{e\}) = \mathbb{C}1$.
- However, these have different monomial semigroups

$$\mathcal{S}_r(\rho) = \{a \in \mathcal{C}_r(\rho) : \mathrm{supp}(a) \text{ is a slice of } G\}$$

($B \subseteq G$ is a slice/bisection if s and r are injective on B).

- Even Better Question: Can we recover G from $(\mathcal{S}_r(\rho), \mathcal{D}_r(\rho))$?

Recovering Groupoids from Semigroups

- Let $\rho : B \twoheadrightarrow G$ be a (strongly) saturated Fell line bundle.
- Problem: Recover G from semigroup structure of

$$(S, D) = (\mathcal{S}_\mathrm{r}(\rho), \mathcal{D}_\mathrm{r}(\rho)).$$

Step 1: Modify the domination relation $<$ on S using D, specifically

$$a < b \quad \Leftrightarrow \quad \exists s \in S \ (as, sa \in D \text{ and } bsa = a = asb)$$

(here $as, sa \in D$ ensures that $a < b$ iff $\mathrm{supp}(a) \Subset \mathrm{supp}(b)$).

Step 2: Show ultrafilters $\mathcal{U}(S)$ form an étale groupoid with product

$$U \cdot V = (UV)^< = \{t > uv : u \in U \text{ and } v \in V\} \text{ when } 0 \notin UV$$

and topology generated by $\mathcal{U}(a) = \{U \in \mathcal{U}(S) : a \in U\}$.

Step 3: Note each $g \in G$ defines $S_g = \{a \in S : g \in \mathrm{supp}(a)\} \in \mathcal{U}(S)$.

Theorem (B.-Clark 2020)

$g \mapsto S_g$ is an étale groupoid isomorphism from G onto $\mathcal{U}(S)$.

- This unifies various groupoid recovery results e.g. by Kumjian (1986), Renault (2008), Exel (2010), Steinberg (2019) and Choi-Gardella-Thiel (2019).

Recovering Groupoids from Semigroups

- What if $\rho : B \twoheadrightarrow G$ is not a line bundle?
- Then we also need to consider the central diagonal

$$\mathcal{Z}_{\mathrm{r}}(\rho) = \{a \in \mathcal{D}_{\mathrm{r}}(\rho) : \forall g \in G^0 \ (a(g) \in \mathbb{C}1_g)\}$$

(each unit fibre B_g has a unit 1_g if ρ is strongly saturated).

- Modify domination $<$ further using $Z = \mathcal{Z}_{\mathrm{r}}(\rho)$, specifically

$$a < b \quad \Leftrightarrow \quad \exists s \in S \ (as, sa \in D,\ bs, sb \in Z \text{ and } bsa = a = asb).$$

- Here $as, sa \in D$ and $bs, sb \in Z$ ensure that

$$a < b \qquad \Leftrightarrow \qquad \mathrm{supp}(a) \Subset b^{-1}[B^\times]$$

(where $B^\times =$ core/invertibles of B).

Theorem

Ultrafilters $\mathcal{U}(S)$ recover G from the semigroup structure of

$$(S, D, Z) = (\mathcal{S}_{\mathrm{r}}(\rho), \mathcal{D}_{\mathrm{r}}(\rho), \mathcal{Z}_{\mathrm{r}}(\rho)).$$

Edging Towards a Duality

- To recover all of $\rho : B \twoheadrightarrow G$, we also need the C*-algebra $\mathcal{C}_\mathrm{r}(\rho)$ and the canonical expectation $\Phi_\mathrm{r} : \mathcal{C}_\mathrm{r}(\rho) \twoheadrightarrow \mathcal{D}_\mathrm{r}(\rho)$.
- To obtain a duality, we further need to isolate sufficiently many distinguishing algebraic properties of quadruples

$$(A, S, Z, \Phi) = (\mathcal{C}_\mathrm{r}(\rho), \mathcal{S}_\mathrm{r}(\rho), \mathcal{Z}_\mathrm{r}(\rho), \Phi_\mathrm{r}),$$

  where $\rho : B \twoheadrightarrow G$ is a strongly saturated Fell bundle, e.g.
  - A is the closure of sums from S and, for all $a \in A$ and $s \in S$,

$$\Phi(s^*as) = s^*\Phi(a)s. \qquad \text{(Normality)}$$

  - S is the closure of compatible sums of dominated elements

$$S^{>} = \{s \in S : \exists t \in S\ (s < t)\}.$$

    ($a, b \in S$ are compatible if $\exists s\ (a < s$ and $bs^*, s^*b \in \mathrm{ran}(\Phi)))$
  - Z is binormal and bistable, i.e. for all $a, b \in S$

$$ab \in Z \quad \Rightarrow \quad aZb \subseteq Z \ \text{ and } \ a\Phi(b), \Phi(a)b \in Z.$$

  - Z is also large in that it contains an approx unit for A.

Structured C*-Algebras

Definition

(A, S, Z, Φ) is a structured C*-algebra if

- A is C*-algebra with faithful normal expectation $\Phi : A \to S$.
- $S = \mathbb{C}S$ is a closed *-subsemigroup generating A, which is itself generated by compatible sums of dominated elements.
- Z is a large binormal bistable central C*-subalgebra of $\mathrm{ran}(\Phi)$.

Example (Dauns-Hofmann-Varela situation)

Any C*-algebra A with large centre $\mathsf{Z}(A)$ (e.g. any unital C*-algebra) defines a structured C*-algebra $(A, A, \mathsf{Z}(A), \mathrm{id}_A)$.

Example (Kumjian-Renault situation)

Any Cartan pair (A, C) defines a structured C*-algebra $(A, \mathsf{N}(C), C, \Phi)$, where Φ is the expectation onto C and

$$\mathsf{N}(C) = \{a \in A : aCa^* + a^*Ca \subseteq C\}.$$

- Key Example: Any strongly saturated Fell bundle $\rho : B \twoheadrightarrow G$ defines a structured C*-algebra $(\mathcal{C}_\mathsf{r}(\rho), \mathcal{S}_\mathsf{r}(\rho), \mathcal{Z}_\mathsf{r}(\rho), \Phi_\mathsf{r})$.

Structured C*-Algebras $\to$ Fell Bundles

- Let (A, S, Z, Φ) be a structured C*-algebra.
- Then the ultrafilters $\mathcal{U}(S)$ form an LCH étale groupoid.
- Each $U \in \mathcal{U}(S)$ also defines a seminorm

$$\|a\|_U = \inf\{\|\Phi(av^*)\| : U \ni u <^* v\}$$

($u <^* v$ means $uv^* \in \mathrm{ran}(\Phi)$, $vv^*, v^*v \in Z$ and $u = uv^*v$).

- In particular, $0_U = \{a \in A : \|a\|_U = 0\}$ is a closed subspace.
- $\bigcup_{U \in \mathcal{U}(S)}(A/0_U)$ forms a strongly saturated Fell bundle $\rho_{\mathcal{U}}$.
- Each $a \in A$ defines a continuous section $\widehat{a} \in \mathcal{C}_{\mathrm{r}}(\rho_{\mathcal{U}})$ by

$$\widehat{a}(U) = a + 0_U.$$

- $a \mapsto \widehat{a}$ is a faithful representation of A onto $\mathcal{C}_{\mathrm{r}}(\rho_{\mathcal{U}})$ with

$$\widehat{S} = \mathcal{S}_{\mathrm{r}}(\rho_{\mathcal{U}}), \quad \widehat{Z} = \mathcal{Z}_{\mathrm{r}}(\rho_{\mathcal{U}}) \quad \text{and} \quad \widehat{\Phi(a)} = \Phi_{\mathrm{r}}(\widehat{a}).$$

Theorem (B. 2021)

Structured C*-algebras are precisely those of the form $(\mathcal{C}_{\mathrm{r}}(\rho), \mathcal{S}_{\mathrm{r}}(\rho), \mathcal{Z}_{\mathrm{r}}(\rho), \Phi_{\mathrm{r}})$ for strongly saturated Fell bundle ρ.

Morphisms

- If (A, S, Z, Φ) and (A', S', Z', Φ') are structured C*-algebras then a structured morphism is a *-homo $\pi : A \to A'$ with

$$\pi[S] \subseteq S', \quad \pi[Z] \subseteq Z' \quad \text{and} \quad \pi(\Phi(a)) = \Phi'(\pi(a)).$$

- Then we get a continuous star-bijective functor $\widehat{\pi}$ from an open subgroupoid of $\mathcal{U}(S')$ to $\mathcal{U}(S)$ defined by

$$\widehat{\pi}(U') = \pi^{-1}[U']^{<} \quad \text{when} \quad \pi^{-1}[U'] \neq \emptyset.$$

- We also get a bundle morphism from the $\widehat{\pi}$-pullback of $\rho_{\mathcal{U}}$ to the ultrafilter bundle $\rho'_{\mathcal{U}}$ of (A', S', Z', Φ').
- In this way, structured morphisms between structured C*-algebras correspond precisely to 'Varela morphisms' between the corresponding strongly saturated Fell bundles.

Theorem (B. 2021)

Under these morphisms, structured C*-algebras and strongly saturated Fell bundles form equivalent categories.