Equivariant twisted KK-theory of noncompact Lie groups

Chi-Kwong Fok

New York University Shanghai

Global Noncommutative Geometry Seminar (Asia-Pacific)

October 25, 2021

Based on joint work with Mathai Varghese.

[FM]

F. and Mathai Varghese,
The ring structure of twisted equivariant KK-theory for noncompact Lie groups
Commun. Math. Phys. **385**, (2021) 633-666, **[1903.05298]**.

Outline of talk

1. Freed-Hopkins-Teleman Theorem and quasi-Hamiltonian manifolds
2. Generalization to noncompact setting using correspondences
3. Baum-Connes and Connes-Kasparov maps
4. Conjectures on representations of loop groups

Twists, loop groups and their representations

Let K be a connected, simple, and simply-connected compact Lie group, viewed as a K-space via the conjugation action.

The equivariant cohomology group $H^3_K(K;\mathbb{Z}) \cong \mathbb{Z}$ classifies the twists for the equivariant K-theory of K, as well as the central extension of the loop group LK by $U(1)$.

Let τ^K_K be a twist corresponding to $m \in \mathbb{Z}$. It can be realized as the Dixmier-Douady bundle $P_eK \times_{L_eK} \mathcal{K}(\mathcal{H}_K^{\otimes m})$, where

1. P_eK and L_eK are based path and based loop groups respectively,
2. $\mathcal{H}_K$ is the basic representation of the level 1 central extension $\widetilde{LK}_1$.

A level ℓ positive energy representation of LK is a representation $\mathcal{H}$ of the level ℓ central extension $\widetilde{LK}_\ell$ where the loop rotation acts with positive weights.

Freed-Hopkins-Teleman Theorem

Theorem (Freed-Hopkins-Teleman)

1. *We have that*

$$K^*_K(K, \tau^K_K)$$

is a ring, with Pontryagin product induced by the multiplication map of K.

2. *Let* $h^\vee$ *be the dual Coxeter number of* K *and* $m \geq h^\vee$. *The pushforward map*

$$\iota^K_! : R(K) \cong K^*_K(pt) \to K^*_K(K, \tau^K_K)$$

induced by the inclusion of the group identity into K *is onto, and its kernel is precisely the level* $m - h^\vee$ *Verlinde ideal* $I_{m-h^\vee}$ *of* $R(K)$. *In other words,* $K^*_K(K, \tau^K_K)$ *is isomorphic to the level* $m - h^\vee$ *Verlinde algebra* $R_{m-h^\vee}(LK)$, *the representation group of level* $m - h^\vee$ *positive energy representations of* LK *with fusion product.*

Freed-Hopkins-Teleman Theorem

$$\begin{aligned}&\text{Irreducible positive energy representations of } LK\\ &\xleftrightarrow{1:1} \Lambda^*_{m-\mathrm{h}^\vee}\\ &:= \{\lambda \in \Lambda^*_+ \,|\, B(\lambda, \alpha_{\mathrm{max}}) \leq m - \mathrm{h}^\vee\}\end{aligned}$$

Note that $W_\mu := \iota_!^K(V_\mu)$ is the irreducible positive energy representation of LK with highest weight $\mu \in \Lambda^*_{m-\mathrm{h}^\vee}$.

Example

$K = SU(2)$. $\mathrm{h}^\vee = 2$.

$$K^*_K(K, \tau^K_K) \cong R_{m-2}(LK) \cong \frac{\mathbb{Z}[\sigma_{\mathrm{std}}]}{(\mathrm{Sym}^{m-1}\sigma_{\mathrm{std}})} \cong \mathbb{Z}^{\oplus m-1}.$$

Geometric cycle description

Generalizing work by Wang and Baum-(Oyono-Oyono)-Schick-Walter, one has that the equivariant twisted K-homology $K_*^K(X, \tau)$, where X is an oriented compact K-manifold, admits a geometric model, and is generated by cycles (M, f, E), where $f : M \to X$ is a K-equivariant map, $E \to M$ a complex vector bundle and M is twisted Spin^c, i.e.,$f^*\tau \otimes \mathrm{Cl}(TM) \cong \mathrm{End}(S)$ for some K-vector bundle S, subject to the three relations

1. K-equivariant cobordism,
2. Disjoint union corresponding to direct sum, and
3. Bundle modification (implementing equivariant Bott periodicity)

Meinrenken showed that the equivariant twisted K-homology $K_*^K(K, \tau_K^K)$, which is isomorphic to $K_K^*(K, \tau_K^K)$ through Poincaré duality, is generated by cycles

$$\{(\mathcal{C}_\mu, \iota, T\mathcal{C}_\mu) | \mu \in \Lambda^*_{m-\mathrm{h}^\vee}\},$$

where $\mathcal{C}_\mu$ is the conjugacy class of K which contains $\exp\left(\frac{B^\sharp(\mu+\rho)}{m}\right)$ and $\iota : \mathcal{C}_\mu \to K$ is the inclusion map.

Quasi-Hamiltonian manifolds

Definition (Alekseev-Malkin-Meinrenken)

A K-manifold N is a quasi-Hamiltonian manifold (or q-Hamiltonian manifold for short) if there exist $\omega_N \in \Omega^2(N)^K$ and an equivariant K-valued map $\mu_N : N \to K$ (called the group-valued moment map) satisfying the following conditions.

1. $d\omega_N + \mu_N^* \eta_K = 0$, where $\eta_K \in \Omega^3(K)$ is the Cartan 3-form $\frac{1}{12} B(\theta^L, [\theta^L, \theta^L])$.
2. Hamiltonian equation, and
3. minimal degeneracy of ω.

q-Hamiltonian manifolds can be thought of as the 'exponentiated' version of ordinary Hamiltonian manifolds.

Example

Conjugacy classes of K with the moment map being inclusion.

Quantization

Any q-Hamiltonian manifold (N, K, ω_N, μ_N) has a canonical twisted Spin^c structure in the sense that there exists $\mathcal{S}$ such that $\mu_N^* \eta_K^{\otimes \mathrm{h}^\vee} \otimes \mathrm{Cl}(TN) \cong \mathrm{End}(\mathcal{S})$. N is said to be quantizable at level $m - \mathrm{h}^\vee$ if there exists $\mathcal{E} \to N$ such that $\mu_N^* \eta_K^{\otimes m - \mathrm{h}^\vee} \cong \mathrm{End}(\mathcal{E})$.

Definition (Meinrenken)

The quantization of (N, K, ω_N, μ_N) is defined to be

$$(N, \mu_N, TN) \in K_*^K(K, \tau_K^K).$$

The noncompact case

Let G be a connected, simply-connected and simple noncompact linear Lie group and K its maximal compact subgroup.

Question: What is the equivariant twisted K-theory of G?

The twists of G are also classified by $H^3_G(G;\mathbb{Z}) \cong \mathbb{Z}$. Let τ^G_G be the twist corresponding to m.

τ^G_G can be realized as $P_eG \times_{L_eG} \mathcal{K}(\mathcal{H}_Q^{\otimes m})$, where Q is a maximal compact subgroup of $G^{\mathbb{C}}$.

Since the conjugation action of G on itself is not proper, $K^*_G(G, \tau^G_G)$ does not make sense. Instead, we consider $KK^G_*(G/K, \Gamma_0(\tau^G_G))$.

The noncompact case

Proposition

The equivariant twisted K-homology $K_*^K(K, \tau_K^K)$ *is naturally isomorphic to* $KK_*^G(G/K, \Gamma_0(\tau_G^G))$.

The isomorphism can be realized using a composition of equivariant Poincaré duality isomorphism, Thom isomorphism, Green-Julg isomorphism and the restriction map restriting G-action to K-action.

What are the generators of $KK_*^G(G/K, \Gamma_0(\tau_G^G))$?

Equivariant twisted correspondences

Generalizing work by Connes-Skandalis and Wang, one can describe the geometric model of $KK_*^G(\Gamma_0(\mathrm{Cl}(TX)), \Gamma_0(\tau_Y))$ using correspondences.

Definition

An G-equivariant twisted correspondence between the G-space X and (Y, τ_Y) is a diagram

$$\begin{array}{ccc} & (M, E) & \\ {\scriptstyle b} \swarrow & & \searrow {\scriptstyle f} \\ X & & Y \end{array} \tag{1}$$

where M is a G-space, $E \to M$ a G-vector bundle, b a proper G-map, f a G-map such that

$$f^*\tau_Y \otimes \mathrm{Cl}(TM) \cong \mathrm{End}(S)$$

for some G-vector bundle S.

Equivariant twisted correspondences

The geometric model $KK^G_{\text{geom}}(\Gamma_0(\text{Cl}(TX)), \Gamma_0(\tau_Y))$ is generated by isomorphism classes of G-equivariant twisted correspondences subject to the same three relations as before (G-equivariant cobordism, disjoint union corresponding to direct sum, and bundle modification).

Theorem

The map

$$\Phi : KK^G_{geom}(\Gamma_0(Cl(TX)), \Gamma_0(\tau_Y)) \to KK^G_*(\Gamma_0(Cl(TX)), \Gamma_0(\tau_Y))$$

which sends the correspondence (1) to

$$b^* \otimes_{\Gamma_0(Cl(TM))} ([[E]] \otimes_{\Gamma_0(Cl(TM))} f_!)$$

is an isomorphism. Here

$$[[E]] \in KK^G_*(\Gamma_0(Cl(TM)), \Gamma_0(Cl(TM))),$$

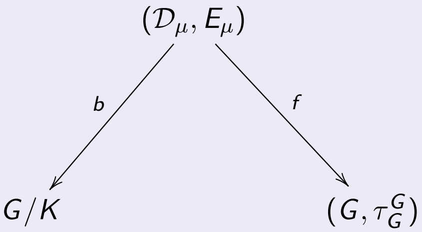
$$b^* \in KK^G_*(\Gamma_0(Cl(TX)), \Gamma_0(Cl(TM))).$$

Equivariant twisted correspondences

Note that $\Gamma_0(\mathrm{Cl}(T(G/K)))$ is Morita equivalent to $C_0(G/K)$ and thus $KK_*^G(\Gamma_0(\mathrm{Cl}(T(G/K))), \Gamma_0(\tau_G^G)) \cong KK_*^G(G/K, \Gamma_0(\tau_G^G))$.

Theorem

$KK_*^G(G/K, \Gamma_0(\tau_G^G))$ is freely generated by the correspondences



(2)

where, $\mu \in \Lambda^*_{k-h^\vee}$, $\mathcal{D}_\mu := G \times_K \mathcal{C}_\mu$, $b : \mathcal{D}_\mu \to G/K$ is the natural projection, $f : \mathcal{D}_\mu \to G$, $[g, c] \mapsto gcg^{-1}$, and $E_\mu := G \times_K T\mathcal{C}_\mu$.

Equivariant twisted correspondences

Definition

We say $g \in G$ is strongly stable if its centralizer G_g is compact. Let $\mathcal{U}$ be the set of strongly stable elements in G.

Remark

$\mathcal{U}$ is open. $\mathcal{U}$ is nonempty iff rk $G =$ rk K. If $\mathcal{U}$ is nonempty, then $\mathcal{U} = \{gtg^{-1} | g \in G, t$ regular element in compact maximal torus $T\}$.

Theorem

Let G and K have the same rank, (M, G, ω_M, μ_M) a q-Hamiltonian manifold with proper G-action and $Im(\mu_M) \in \mathcal{U}$. Suppose $\mu_M : M \to G$ composed with $p : G \to G/K$ has eK as a regular value, and let $N = \mu_M^{-1}(K)$. Then

1. *$(N, K, i^*\omega_M, \mu_M \circ i)$ is a q-Hamiltonian K-manifold.*
2. *$M \cong G \times_K N$ G-equivariantly and $\mu_M([g, n]) = Ad_g \mu_n(n)$.*

Equivariant twisted correspondences

Definition

The quantization of (M, G, ω_M, μ_M) satisfying the quantizability condition and conditions in the previous theorem is defined to be the equivariant twisted correspondence in $KK_*^G(G/K, \Gamma_0(\tau_G^G))$

$$\begin{array}{ccc} & (M, G \times_K TN) & \\ {}^{b}\swarrow & & \searrow^{f} \\ G/K & & G \end{array}$$

where $b : M \cong G \times_K N \to G/K$ is the natural projection and $f = \mu_M$.

Baum-Connes and Connes-Kasparov maps

We also consider $K_*(\Gamma_0(\tau_G^G) \rtimes G)$, and the following diagram

$$\begin{array}{ccc} K_K^*(\mathrm{pt}) & \xrightarrow{\text{D-Ind}} & K_*(C_r^* G) \\ \downarrow \iota_!^K & & \downarrow \iota_!^G \\ K_K^*(K, \tau_K^K) & \xrightarrow[q]{} & K_*(\Gamma_0(\tau_G^G) \rtimes G) \end{array}$$

where q is the composition of the following maps

$$\begin{aligned} KK_*^K(\mathrm{pt}, \Gamma(\tau_K^K)) &\xrightarrow{[G/K]\otimes\cdot} KK_*^K(G/K, \Gamma(\tau_K^K)) \\ &\xrightarrow{\cdot\otimes[i]} KK_*^K(G/K, \Gamma_0(\tau_K^G)) \\ &\xrightarrow{(\mathrm{res}_G^K)^{-1}} KK_*^G(G/K, \Gamma_0(\tau_G^G)) \\ &\xrightarrow{\mathrm{BC}} K_*(\Gamma_0(\tau_G^G) \rtimes G) \end{aligned}$$

Baum-Connes and Connes-Kasparov maps

The composition

$$K_*(\Gamma_0(\tau_K^G) \rtimes K) \xrightarrow{\text{Green-Julg}} K_K^*(G, \tau_K^G) \longrightarrow K_K^*(K, \tau_K^K) \xrightarrow{q} K_*(\Gamma_0(\tau_G^G) \rtimes G)$$

is the Connes-Kasparov map.

Theorem

1. *The diagram*

$$\begin{array}{ccc} K_K^*(pt) & \xrightarrow{D\text{-}Ind} & K_*(C_r^* G) \\ \downarrow{\scriptstyle \iota_!^K} & & \downarrow{\scriptstyle \iota_!^G} \\ K_K^*(K, \tau_K^K) & \xrightarrow[q]{} & K_*(\Gamma_0(\tau_G^G) \rtimes G) \end{array}$$

   commutes.

2. *The Baum-Connes map*

$$KK_*(G/K, \Gamma_0(\tau_G^G)) \to K_*(\Gamma_0(\tau_G^G) \rtimes G)$$

   is an isomorphism and so is q.

3. *The Connes-Kasparov map is an isomorphism.*

Proof sketch

We apply Mayer-Vietoris sequence for K-theory groups with respect to a finite G-invariant closed cover $\{\mathcal{F}_i\}$ of G such that the Baum-Connes map restricted to any $\mathcal{F}_i$ and their intersection is an isomorphism.

We shall find $\{\mathcal{F}_i\}$ such that each $\mathcal{F}_i$ and their intersection are stratifications of finitely many strata $\mathcal{O}_j$, and each stratum is

1. either a finite disjoint union of fiber bundles over some cones on which G acts trivially, or
2. a finite disjoint union of conjugacy classes.

Baum-Connes isomorphism holds trivially for fiber bundles over cones as all K-groups vanish.

Proof sketch

Baum-Connes isomorphism also holds for any conjugacy class $\mathcal{C} \cong G/H$.

$$\begin{array}{ccc} KK_*^G(G/K, \tau_G^G|_{\mathcal{C}}) & \longrightarrow & K_*(\Gamma_0(\tau_G^G|_{\mathcal{C}}) \rtimes G) \\ \downarrow \cong & & \downarrow \cong \\ KK_*^H(G/K, \tau_G^G|_{\{g\}}) & \longrightarrow & K_*(\tau_G^G|_{\{g\}} \rtimes H) \end{array}$$

It all boils down to the bottom map, which is the Baum-Connes map for H-action with coefficient $\mathcal{K}(\mathcal{H})$, which is an isomorphism by Chabert-Echteroff-Nest.

The closed cover $\{\mathcal{F}_i\}$ can be explicitly constructed using the Steinberg map

$$\begin{aligned} \mathrm{St} : G &\to \mathbb{C}^n \\ g &\mapsto (\chi_1(g), \cdots, \chi_n(g)) \end{aligned}$$

which is conjugation invariant. Each fiber is a stratification of conjugacy classes, and the base Im(St) can be stratified according to 'genericity'.

Representations of LG

Recall the Connes-Kasparov map

$$\begin{aligned} \text{D-Ind} : K_K^*(\text{pt}) &\to K_*(C_r^*(G)) \\ V_\mu &\mapsto \text{Ind}(\partial\!\!\!/_{G/K} \otimes V_\mu), \end{aligned}$$

where $\partial\!\!\!/_{G/K} \otimes V_\mu$ acts on $L^2(G/K, G \times_K (S \otimes V_\mu))$, is an isomorphism, and $K_*(C_r^*(G))$ gives the tempered representations of G. In particular, if G and K are of the same rank, then it gives the discrete series representations and their limits.

Any irreducible tempered representation of G can be obtained from induction of a representation of a cuspidal parabolic subgroup $P = MAN$ associated to a discrete series representation or a limit of discrete series representation theory of M.

Representations of LG

Recall the commutative diagram

$$\begin{array}{ccc} K_K^*(\mathrm{pt}) & \xrightarrow{\text{D-Ind}} & K_*(C_r^* G) \\ \downarrow \iota_!^K & & \downarrow \iota_!^G \\ K_K^*(K, \tau_K^K) & \xrightarrow[q]{} & K_*(\Gamma_0(\tau_G^G) \rtimes G) \end{array}$$

Conjecture

1. *The K-group $K_*(\Gamma_0(\tau_G^G) \rtimes G)$ gives the tempered representations of LG in a suitable sense.*
2. *The map q is given by the Dirac induction $W_\mu \mapsto Ind(\partial\!\!\!/_{LG/K} \otimes V_\mu)$ for $\mu \in \Lambda^*_{m-h^\vee}$.*

Representations of LG

Example

Let G be a complex simple Lie group. Let $\beta_\mu \in K_*(C_r^* G)$ be the K-theory element associated with the unitary principal series representation of G induced from a discrete series or a limit of discrete series representation of M with weight μ. For $\mu \in \Lambda^*_{m-\mathrm{h}^\vee}$,

$$\begin{array}{ccc} V_\mu & \xrightarrow{\text{D-Ind}} & \beta_{\mu+\rho} \\ \Big\downarrow{\scriptstyle \iota_!^K} & & \Big\downarrow{\scriptstyle \iota_!^G} \\ W_\mu & \xrightarrow[q]{} & ?\gamma_{\mu+\rho} \end{array}$$

$K_K^*(K, \tau_K^K) \cong \mathbb{Z}^{\Lambda^*_{m-\mathrm{h}^\vee}}$. $K_*(\Gamma_0(\tau_G^G) \rtimes G) \cong \mathbb{Z}^{\Lambda^*_{m-\mathrm{h}^\vee}+\rho}$.