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# Superrigidity for dense subgroups of Lie groups and their actions on homogeneous space

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# Main result

Consider the group of isometries $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

**Theorem (Drimbe - V, 2021)**

Let $\mathcal{S}$ be a finite set of prime numbers and $\Gamma = \mathrm{PSL}(2,\mathbb{Z}[\mathcal{S}^{-1}])$. Consider $M = L^\infty(\mathbb{H}^2) \rtimes \Gamma$.

- (well known) If $|\mathcal{S}| = 0$, then $M \cong L^\infty([0,1]) \mathbin{\overline{\otimes}} B(H)$.
- If $|\mathcal{S}| = 1$, then $M$ is a nonamenable factor and $M \cong L^\infty(Y) \rtimes \Lambda$ for uncountably many nonisomorphic $\Lambda$.
- If $|\mathcal{S}| \geq 2$, we have W$^*$-superrigidity: if $M \cong L^\infty(Y) \rtimes \Lambda$, we essentially have $\Lambda \cong \Gamma$ and $\Lambda \curvearrowright Y$ conjugate with $\Gamma \curvearrowright \mathbb{H}^2$.

# W*-equivalence and orbit equivalence

We consider group actions $\Gamma \curvearrowright (X, \mu)$ that are (essentially) free, ergodic and nonsingular.

Two such actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are

- **conjugate** if there exists a group isomorphism $\delta : \Gamma \to \Lambda$ and a nonsingular isomorphism $\Delta : X \to Y$ such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$.
- **orbit equivalent** if there exists a nonsingular isomorphism $\Delta : X \to Y$ such that $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$.
- **W*-equivalent** if $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$.

**The strongest form of rigidity**

We say that $\Gamma \curvearrowright (X, \mu)$ is **W*-superrigid** if any W*-equivalent action $\Lambda \curvearrowright Y$ must be conjugate to $\Gamma \curvearrowright X$. (Correct notion: deal with amplifications.)



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# W*-superrigidity for Bernoulli actions

$\rightsquigarrow$ Popa's deformation/rigidity theory.

**Theorem (Popa 2003, Ioana 2010, Ioana-Popa-V 2010)**

Let $\Gamma$ be an infinite group without finite normal subgroups. Let $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ be a Bernoulli action.

Assume that $\Gamma$ has property (T), or that $\Gamma = \Gamma_1 \times \Gamma_2$ with $\Gamma_1$ infinite and $\Gamma_2$ nonamenable.

Then, $\Gamma \curvearrowright (X, \mu)$ is W*-superrigid.

- **Deformation (Popa's malleability) :** $\alpha_t \in \mathrm{Aut}(X \times X)$, commuting with the diagonal $\Gamma$-action given by $g \cdot (x, y) = (g \cdot x, g \cdot y)$, and $\alpha_0 = \mathrm{id}$ and $\alpha_1 = \mathrm{flip}$.
- **Rigidity :** property (T) or spectral gap.

# A systematic approach to W*-superrigidity for $\Gamma \curvearrowright (X, \mu)$

1. Prove that $L^\infty(X) \rtimes \Gamma$ has a **unique Cartan subalgebra** up to unitary conjugacy.
2. Prove that $\Gamma \curvearrowright X$ is orbit equivalence (OE) superrigid :

   any orbit equivalent action $\Lambda \curvearrowright Y$ must be conjugate to $\Gamma \curvearrowright X$.

- Assume that $\Lambda \curvearrowright Y$ is orbit equivalent with $\Gamma \curvearrowright X$.
  So, we have $\Delta : X \to Y$ with $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$.
- Define the **Zimmer 1-cocycle** $\omega : \Gamma \times X \to \Lambda : \Delta(g \cdot x) = \omega(g, x) \cdot \Delta(x)$.
- We have $\omega(gh, x) = \omega(g, h \cdot x)\, \omega(h, x)$.
- We say that $\omega \sim \omega'$ are **cohomologous 1-cocycles** if there exists $\varphi : X \to \Lambda$ such that $\omega'(g, x) = \varphi(g \cdot x)\, \omega(g, x)\, \varphi(x)^{-1}$.
- Every group homomorphism $\delta : \Gamma \to \Lambda$ defines a 1-cocycle $\omega(g, x) = \delta(g)$.


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# Popa's cocycle superrigidity theorem for Bernoulli actions

## Cocycle superrigidity

We say that a nonsingular action $\Gamma \curvearrowright (X, \mu)$ is **cocycle superrigid** if every 1-cocycle $\omega : \Gamma \times X \to \Lambda$ with values in a countable group $\Lambda$ is cohomologous to a group homomorphism $\delta : \Gamma \to \Lambda$.

**Essentially true:** if $L^\infty(X) \rtimes \Gamma$ has a unique Cartan and if $\Gamma \curvearrowright (X, \mu)$ is cocycle superrigid, then $\Gamma \curvearrowright (X, \mu)$ is W*-superrigid.

## Popa 2005 & 2006

Let $\Gamma$ be an infinite group either with property (T),
or $\Gamma = \Gamma_1 \times \Gamma_2$ with $\Gamma_1$ infinite and $\Gamma_2$ nonamenable.

Then every Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ is cocycle superrigid.

# Popa's cocycle superrigidity theorem for Bernoulli actions

**Popa 2005 & 2006**

Let $\Gamma$ be an infinite group either with property (T),
or $\Gamma = \Gamma_1 \times \Gamma_2$ with $\Gamma_1$ infinite and $\Gamma_2$ nonamenable.

Then every Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ is cocycle superrigid.

- Let $\omega : \Gamma \times X \to \Lambda$ be a 1-cocycle.
- Diagonal $\Gamma \curvearrowright X \times X$ and 1-cocycles $\omega_0(g, x, y) = \omega(g, x)$ and $\omega_1(g, x, y) = \omega(g, y)$.
- Let $\alpha_t \in \mathrm{Aut}(X \times X)$ be the malleable deformation.

  $\rightsquigarrow$ We get 1-cocycles $\omega_t : \Gamma \times (X \times X) \to \Lambda : \omega_t(g, x, y) = \omega_0(g, \alpha_t(x, y))$.
- Spectral gap ensures that $\omega_t \sim \omega_s$ if $|t - s|$ is small. Thus $\omega_0 \sim \omega_1$. Thus $\omega \sim \delta$.



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# Ioana's cocycle superrigidity for profinite actions

**A special case of Ioana's theorem (2008)**

Let $K$ be a totally disconnected compact group. Let $\Gamma \subset K$ be a countable dense subgroup.

If $\Gamma$ has property (T), then the translation action $\Gamma \curvearrowright K$ is virtually cocycle superrigid.

**Virtual:** every 1-cocycle $\omega : \Gamma \times K \to \Lambda$ is cohomologous to a group homomorphism after restriction to $\Gamma_0 \times K_0$, where $K_0 \subset K$ is an open subgroup and $\Gamma_0 = \Gamma \cap K_0$.

**Example:** $\mathrm{SL}(n, \mathbb{Z}) \subset \mathrm{SL}(n, \mathbb{Z}_p)$ for $n \geq 3$.

- Let $\omega : \Gamma \times K \to \Lambda$ be a 1-cocycle.
- For every $k \in K$, we have a 1-cocycle $\omega_k : \Gamma \times K \to \Lambda : \omega_k(g, x) = \omega(g, xk)$.
- By property (T), we have $\omega_k \sim \omega$ for all $k$ in a neighborhood of $e$ in $K$.
- Thus $\omega_k \sim \omega$ for all $k \in K_0$ where $K_0$ is an open subgroup of $K$.

# Cocycle superrigidity for infinite measure preserving actions

**Theorem (Popa-V, 2008)**

The natural action $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathbb{R}^n$ with $n \geq 5$ is cocycle superrigid.

**Deformation:** the rotation of $\mathbb{R}^n \times \mathbb{R}^n$ that commutes with the diagonal action.

**Rigidity:** only when $n \geq 5$, the diagonal action of $\mathrm{SL}(n,\mathbb{Z})$ on $\mathbb{R}^n \times \mathbb{R}^n$ has property (T).

# Back to isometries of the hyperbolic plane

We consider $\Gamma \subset \mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$. Note that $\mathbb{H}^2 = \mathrm{PSL}(2,\mathbb{R})/\,\mathrm{PSO}(2,\mathbb{R})$.

**General setup.**

- Dense subgroup $\Gamma$ of a locally compact second countable (lcsc) group $G$.
- Closed subgroup $P \subset G$.
- We consider the translation action $\Gamma \curvearrowright G/P$.

$\rightsquigarrow$ Does $L^\infty(G/P) \rtimes \Gamma$ have a unique Cartan subalgebra, up to unitary conjugacy ?

$\rightsquigarrow$ Is the action $\Gamma \curvearrowright G/P$ cocycle superrigid ?

# Cocycle superrigidity for dense subgroups

**Definition (Drimbe-V, 2021)**

A dense subgroup $\Gamma$ of a lcsc group $G$ is **essentially cocycle superrigid** if for every 1-cocycle $\omega : \Gamma \times G \to \Lambda$ for the translation action $\Gamma \curvearrowright G$, there exists

- an open subgroup $G_0 \subset G$,
- a covering $\pi : G_1 \to G_0$,
- such that, with $\Gamma_1 = \pi^{-1}(\Gamma \cap G_0)$, the 1-cocycle
$$\omega_1 : \Gamma_1 \times G_1 \to \Lambda : \omega_1(g, x) = \omega(\pi(g), \pi(x))$$
is cohomologous to a group homomorphism.

**Proposition.** If $G$ is a connected Lie group and $P \subset G$ is a closed subgroup such that $\pi_1(P) \to \pi_1(G)$ is surjective, then essential cocycle superrigidity of $\Gamma \subset G$ implies plain cocycle superrigidity for $\Gamma \curvearrowright G/P$.

# A cocycle superrigidity theorem for dense subgroups

**Theorem (Drimbe-V, 2021)**

Let $G$ and $H$ be lcsc groups with $H$ compactly generated. Let $\Gamma \subset G \times H$ be a lattice.

Assume that $H = H_1 \times H_2$ such that $H_1 \curvearrowright (G \times H)/\Gamma$ is **ergodic** and $H_2 \curvearrowright (G \times H)/\Gamma$ is **strongly ergodic**.

Then, the projection $\Gamma_0 \subset G$ of $\Gamma$ is essentially cocycle superrigid.

**Note:** for $G$ compact and totally disconnected, this is Drimbe-Ioana-Peterson (2019).

**Prototype example:** $\Gamma = \mathrm{SL}(2, \mathbb{Z}[p^{-1}, q^{-1}]) \subset \mathrm{SL}(2, \mathbb{R})$ is essentially cocycle superrigid.

- The diagonal embedding $\Gamma \subset \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{Q}_p) \times \mathrm{SL}(2, \mathbb{Q}_q)$ is a lattice.
- (Gelbart-Jacquet, 1978, work on automorphic representations) The unitary representation of $\mathrm{SL}(2, \mathbb{Q}_p)$ on $L^2_0((\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{Q}_p) \times \mathrm{SL}(2, \mathbb{Q}_q))/\Gamma)$ does not weakly contain the trivial representation.



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# Cocycle superrigid dense subgroups

**Corollary (Drimbe-V 2021, using Clozel 2002 to get spectral gap)**

The following dense subgroups are essentially cocycle superrigid.

- $\mathrm{SL}(n, \mathbb{Z}[\mathcal{S}^{-1}]) \subset \mathrm{SL}(n, \mathbb{R})$ if $(n-1)\,|\mathcal{S}| \geq 2$. Here $\mathcal{S}$ is a set of prime numbers.
- $\mathrm{SL}(n, \mathcal{O}_K) \subset SL(n, \mathbb{R})$ if $(n-1)(a+b-1) \geq 2$.

  Here $\mathbb{Q} \subset K \subset \mathbb{R}$ is a real algebraic number field, with ring of integers $\mathcal{O}_K$, with $a$ real embeddings and $b$ complex embeddings.
- Similar subgroups of $\mathrm{SO}^+(n, m, \mathbb{R})$. More general $S$-arithmetic groups.

⇝ But $\mathrm{SL}(2, \mathbb{Z}[1/p]) \subset \mathrm{SL}(2, \mathbb{R})$ does not satisfy any cocycle superrigidity.

Same for $\mathrm{SL}(2, \mathbb{Z}[\sqrt{N}]) \subset \mathrm{SL}(2, \mathbb{R})$.

# Key points of the proof

**Theorem (Drimbe-V, 2021)**

Let $G$ and $H$ be lcsc groups. Let $\Gamma \subset G \times H$ be a lattice.

Assume that $H = H_1 \times H_2$ such that $H_1 \curvearrowright (G \times H)/\Gamma$ is **ergodic** and $H_2 \curvearrowright (G \times H)/\Gamma$ is **strongly ergodic**. Then, the projection $\Gamma_0 \subset G$ of $\Gamma$ is essentially cocycle superrigid.

Write $\mathrm{pr} : G \times H \to G$. Thus, $\Gamma_0 = \mathrm{pr}(\Gamma)$.

**Remark:** for a lcsc group $\mathcal{G}$ and closed subgroup $L \subset \mathcal{G}$, every 1-cocycle for $L \curvearrowright \mathcal{G}$ is trivial.

- View $\Gamma$ and $\Gamma_0$ as acting by **right** translation on resp. $G \times H$ and $G$.
- A 1-cocycle $\omega : \Gamma_0 \times G \to \Lambda$ is "the same as" a 1-cocycle for $H \times \Gamma \curvearrowright G \times H$,
- which is "the same as" a 1-cocycle $\omega' : H \times (G \times H)/\Gamma \to \Lambda$.
- For every $g \in G$, we have a 1-cocycle $\omega'_g(h, (x, k)\Gamma) = \omega'(h, (gx, k)\Gamma)$.
- By rigidity, $\omega'_g \sim \omega'$ if $g$ is close to $e$ ...... conclusion.

# About uniqueness of Cartan subalgebras

**Question:** given a free ergodic nonsingular $\Gamma \curvearrowright (X,\mu)$, is $L^\infty(X)$ the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$, up to unitary conjugacy ?

$\rightsquigarrow$ (Ozawa-Popa 2007) Yes, if $\Gamma = \mathbb{F}_n$ and $\Gamma \curvearrowright (X,\mu)$ is a profinite action.

**Theorem (Popa-V, 2011)**

When $\mathbb{F}_n \curvearrowright (X,\mu)$ is **any** free ergodic pmp action, $L^\infty(X) \rtimes \mathbb{F}_n$ has a unique Cartan.

$\rightsquigarrow$ Since then, this **Cartan-rigidity** property has been proven for several families of groups.

We are looking at infinite measure preserving actions $\Gamma \curvearrowright \mathbb{H}^2 = \mathrm{PSL}(2,\mathbb{R}) / \mathrm{PSO}(2,\mathbb{R})$.

$\rightsquigarrow$ Another point of view is needed. Such an action could in principle even be amenable.



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# About uniqueness of Cartan subalgebras

Let $\mathcal{S}$ be a finite set of prime numbers. Put $\Gamma = \mathrm{PSL}(2, \mathbb{Z}[\mathcal{S}^{-1}])$. Consider $\Gamma \curvearrowright \mathbb{H}^2$.

- $\Gamma$ is a lattice in $G \times H$ with $G = \mathrm{PSL}(2, \mathbb{R})$ and $H = \prod_{p \in \mathcal{S}} \mathrm{PSL}(2, \mathbb{Q}_p)$.
- We have $\mathbb{H}^2 = G/K$ with $K = \mathrm{PSO}(2, \mathbb{R})$.
- Then, the $\mathrm{II}_\infty$ factors $L^\infty(\mathbb{H}^2) \rtimes \Gamma$ and $L^\infty\big(\Gamma \backslash (G \times H)/K\big) \rtimes H$ are canonically isomorphic.

**Theorem (Brothier-Deprez-V, 2017)**

The locally compact group $H$ is Cartan-rigid: for every essentially free, ergodic, **pmp** action $H \curvearrowright X$, the $\mathrm{II}_\infty$ factor $L^\infty(X) \rtimes H$ has a unique Cartan subalgebra up to unitary conjugacy.

$\rightsquigarrow$ **Attention:** $L^\infty(X)$ is not a Cartan subalgebra in $L^\infty(X) \rtimes H$. One needs to use **cross section equivalence relations**.

# Back to the main result

**Theorem (Drimbe - V, 2021)**

Let $\mathcal{S}$ be a finite set of prime numbers and $\Gamma = \mathrm{PSL}(2, \mathbb{Z}[\mathcal{S}^{-1}])$. Consider $M = L^\infty(\mathbb{H}^2) \rtimes \Gamma$.

- (well known) If $|\mathcal{S}| = 0$, then $M \cong L^\infty([0,1]) \mathbin{\overline{\otimes}} B(H)$.
- If $|\mathcal{S}| = 1$, then $M$ is a nonamenable factor and $M \cong L^\infty(Y) \rtimes \Lambda$ for uncountably many nonisomorphic $\Lambda$.
- If $|\mathcal{S}| \geq 2$, we have W*-superrigidity: if $M \cong L^\infty(Y) \rtimes \Lambda$, we essentially have $\Lambda \cong \Gamma$ and $\Lambda \curvearrowright Y$ conjugate with $\Gamma \curvearrowright \mathbb{H}^2$.

- The first point is trivial: $\mathrm{PSL}(2, \mathbb{Z}) \curvearrowright \mathbb{H}^2$ admits a fundamental domain.
- The proof of the third point has been sketched.

# Absence of rigidity

What happens with $\Gamma = \mathrm{PSL}(2, \mathbb{Z}[1/p]) \curvearrowright \mathbb{H}^2$ ?

- The orbit equivalence relation turns out to be **treeable**.

$\rightsquigarrow$ One can choose $X \subset \mathbb{H}^2$ of positive, finite measure, and one can choose a measure preserving free action $\mathbb{F}_2 \curvearrowright X$ such that $(\Gamma \cdot x) \cap X = \mathbb{F}_2 \cdot x$ for all $x \in X$.

- Then, $L^\infty(\mathbb{H}^2) \rtimes \Gamma \cong (L^\infty(X) \rtimes \mathbb{F}_2) \mathbin{\overline{\otimes}} B(H)$.
- Then, $L^\infty(\mathbb{H}^2) \rtimes \Gamma \cong L^\infty(\mathbb{H}^2) \rtimes (\Lambda_1 * \Lambda_2 \times \mathbb{Z})$ for a well chosen action, with $\Lambda_1, \Lambda_2$ arbitrary infinite amenable groups.

# Absence of rigidity: treeability

**Definition (Adams, 1988)**

A countable nonsingular equivalence relation $\mathcal{R}$ on $(X, \mu)$ is called **treeable** if there exists a **treeing:**

a Borel set $\mathcal{T} \subset \mathcal{R}$ such that, when viewing $\mathcal{T}$ as the edges of a graph with vertex set $X$, for a.e. $x \in X$, the restriction of $\mathcal{T}$ to the equivalence class $\mathcal{R} \cdot x$ is a tree.

**Obvious examples:** the orbit equivalence relation of a free nonsingular action $\mathbb{F}_n \curvearrowright (X, \mu)$ is treeable.

**Theorem (Hjorth, 2005)**

If $\mathcal{R}$ is a nonamenable treeable equivalence relation on $(X, \mu)$ and if $\mu$ is a finite/infinite invariant measure, we may choose a Borel set $Y \subset X$ and a free action $\mathbb{F}_n \curvearrowright Y$ with $2 \leq n \leq +\infty$ such that $(\mathcal{R} \cdot x) \cap Y = \mathbb{F}_n \cdot y$ for all $y \in Y$.

# Treeability of locally compact groups

**Definition (Conley, Gaboriau, Marks & Tucker-Drob, 2021)**

A lcsc group $H$ is called **strongly treeable** if for every free pmp action $H \curvearrowright (X, \mu)$, the cross section equivalence relation is treeable.

**Cross section:** a Borel set $Y \subset X$ such that for a neighborhood $\mathcal{U} \subset H$ of $e$, the map $\mathcal{U} \times Y \to X : (h, y) \mapsto h \cdot y$ is injective and such that $H \cdot \mathcal{U}$ is conull.

**Cross section equivalence relation:** the countable equivalence relation $\mathcal{R}$ given $\mathcal{R} = \{(y, z) \in Y \times Y \mid y \in H \cdot z\}$.

- **Easy example:** $\mathrm{PSL}(2, \mathbb{Q}_p)$ is strongly treeable. Thus, $\mathrm{SL}(2, \mathbb{Z}[1/p]) \curvearrowright \mathbb{H}^2$ is treeable and has no rigidity.
- **Theorem (CGMT 2021):** $\mathrm{PSL}(2, \mathbb{R})$ is strongly treeable. Thus, $\mathrm{SL}(2, \mathbb{Z}[\sqrt{N}]) \curvearrowright \mathbb{H}^2$ is treeable and has no rigidity.