Some results on continuous orbit equivalence rigidity

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Overview

(Part of this talk is based on joint work with Nhan-Phu Chung)

Convention

G, H: countably infinite discrete groups;

$G \curvearrowright X$: a continuous action on a compact Hausdorff space X.

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Remark

Many questions/theorems have counterparts in the measurable setting, e.g. for probability measure preserving actions, and we put emphasis on these analogies.

Two equivalence relations

Given two topologically free actions: $G \overset{\cdot}{\curvearrowright} X$ and $H \overset{*}{\curvearrowright} Y$.

(1) Conjugacy: $\exists\ \phi : G \cong H$ and $\theta : X \approx Y$ s.t.
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(2) Continuous orbit equivalence: $\exists\ \theta : X \approx Y$ and continuous maps
$c : G \times X \to H,\ c' : H \times Y \to G$ s.t.

$$\theta(g \cdot x) = c(g, x) * \theta(x), \quad \theta^{-1}(h * y) = c'(h, y) \cdot \theta^{-1}(y);$$

i.e. $\theta(G \cdot x) = H * \theta(x), \qquad \theta^{-1}(H * y) = G \cdot \theta^{-1}(y).$

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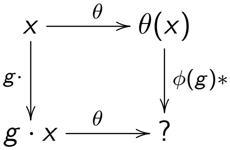

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i.e. $\quad \theta(G \cdot x) = H * \theta(x), \qquad \theta^{-1}(H * y) = G \cdot \theta^{-1}(y).$

Clearly, (1) $\Rightarrow$ (2). In general (2) $\not\Rightarrow$ (1).

Historical remarks

(1) Special cases and weaker versions of coe have been studied by Boyle, Giordano, Matsumoto, Matui, Putnam, Skau, Suzuki, Tomiyama et. al.

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(2) Lots of work has been done in the measurable setting, i.e. studying OE. See e.g. the famous work of H. Dye, Ornstein-Weiss, Zimmer, Furman, Gaboriau, Monod-Shalom, Popa, Kida, Ioana et. al.

There are also related notions, e.g. integral OE, L^p-OE, Shannon-OE, ME etc.

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Definition (Continuous cocycles)

$c : G \times X \to H$ is a *continuous cocycle* if c is continuous and $c(g_1g_2, x) = c(g_1, g_2x)c(g_2, x)$ for all $g_1, g_2 \in G$ and all $x \in X$.

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Clearly, if $\phi \in Hom(G, H)$, then $c(g, x) := \phi(g)$ defines a continuous cocycle.

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RK: In the measurable setting, we have Popa's famous OE superrigidity thm.

Popa's OE superrigidity thm

Theorem (Popa 2005)

Let G be a group with property (T) but contains no non-trivial finite normal subgroups, e.g. $G = PSL_n(\mathbb{Z})(n \geq 3)$. If a Bernoulli shift $G \curvearrowright (X_0, \mu_0)^G$ is OE to a free ergodic p.m.p. action $H \curvearrowright (Y, \nu)$, then they are (measurably) conjugate.

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Remarks:

(1) By Ornstein-Weiss thm, any two non-atomic, p.m.p. ergodic actions of any infinite amenable groups are OE.

(2) Many other groups are known to satisfy the above thm by the work of Popa, Peterson–Sinclair, Ioana–Tucker-Drob et.al.. Moreover, some other actions are also known to satisfy the above thm by the work of Popa, Ioana, Furman, Gaboriau–Ioana–Tucker-Drob, Drimbe et.al..

Why studying coe?

One motivation comes from the connection between coe and C^*-algebras.

Given $G \curvearrowright X$, we can construct a C^*-algebra $C(X) \rtimes_r G$.

For two actions $G \curvearrowright X$ and $H \curvearrowright Y$, we have:

$$\boxed{G \curvearrowright X \sim_{conj.} H \curvearrowright Y}$$

$$\Rightarrow \boxed{G \curvearrowright X \sim_{coe} H \curvearrowright Y} \Rightarrow \boxed{C(X) \rtimes_r G \cong C(Y) \rtimes_r H}.$$

Theorem (X. Li 2015)

Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free actions. TFAE.
(1) $G \curvearrowright X \sim_{coe} H \curvearrowright Y$;
(2) there is a C^*-isomorphism $\Phi : C(X) \rtimes_r G \cong C(Y) \rtimes_r H$ with $\Phi(C(X)) = C(Y)$.

RK: in the measurable setting, OE has a similar characterization using crossed product vN algebras due to Singer 1955.

How to prove coe rigidity?

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Definition (Trivial cocycles)

Let $c_i : G \times X \to H$ be continuous cocycles.

- $c_1 \sim c_2$ if $c_1(g,x) = t(gx)c_2(g,x)t(x)^{-1}$ for some continuous map $t : X \to H$, which is called a *transfer map*.
- $c : G \times X \to H$ is a *trivial cocycle* if $c \sim \phi \in Hom(G,H)$ for some ϕ.

Proposition (X. Li)

If $\phi : G \overset{\cdot}{\curvearrowright} X \sim_{coe} H \overset{*}{\curvearrowright} Y$ and $c \sim \rho \in Iso(G,H)$ via a transfer map t, then $X \ni x \overset{\theta}{\mapsto} t(x)^{-1} * \phi(x) \in Y$ and ρ give rise to a conjugacy between the two actions. When G is torsion-free and amenable, then we may just require $\rho \in Hom(G,H)$, a priori.

Let us verify $\theta(x) = t(x)^{-1} * \phi(x)$ is equivariant:

$$\begin{array}{ccccc} x & \in & X \xrightarrow{\theta} Y & \ni & t(x)^{-1} * \phi(x) \\ & & g\cdot \big\downarrow \qquad \big\downarrow \rho(g)* & & \\ g \cdot x & \in & X \xrightarrow{\theta} Y & \ni & ? \end{array}$$

The diagram commutes means that at the place ?, we have

$$t(g \cdot x)^{-1} * \phi(g \cdot x) = \theta(g \cdot x) = \rho(g) * \theta(x) = [\rho(g)t(x)^{-1}] * \phi(x),$$

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which indeed holds since ϕ gives the coe and $c \sim \rho$ via t, i.e.

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RK: A similar criterion is known in the measurable setting. So roughly speaking, to show coe $\Rightarrow$ conjugacy, the main task is to untwist the orbit cocycles, i.e. show the orbit cocycles are trivial.

Known coe rigidity results

For the following $G \curvearrowright X$, $G \curvearrowright X \sim_{coe} H \curvearrowright Y$ implies they are conjugate for any top. free $H \curvearrowright Y$.

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1. (K. Schmidt 1995) $\mathbb{Z}^{d\geq 2} \curvearrowright A^{\mathbb{Z}^d}$: the full shift action, A is any finite set.
2. (Boyle–Tomiyama 1998) $\mathbb{Z} \curvearrowright X$: any top. free and top. transitive action and if $H = \mathbb{Z}$.
3. (X. Li 2015) $G \curvearrowright X_0^G$, where X_0 is cpt with $|X_0| > 1$ and G is a solvable duality group; and also for some subshifts.
4. (Chung-J. 2016, Cohen 2017) $G \curvearrowright A^G$: the full shift action and G is any f.g. group with one end. (A topological version of Popa's OE superrigidity thm)
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RK: only Boyle–Tomiyama's result applies for minimal $G \curvearrowright X$.

Other related results

Theorem (Cortez-Medynets 2016)

For two minimal equicontinuous $\mathbb{Z}^d$-actions, coe implies the two actions are virtually piecewisely conjugate.

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RK: the above can be viewed as a topological version of Ioana's OE superrigidity thm for profinite actions (in the measurable setting).

Main results

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Do we have coe rigidity for infinite virtually cyclic group actions?

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Extending Boyle-Tomiyama's result, we prove

Theorem (J. 2021)

Let D_∞ be the infinite dihedral group, i.e. $D_\infty = \mathbb{Z} \rtimes \frac{\mathbb{Z}}{2\mathbb{Z}} = \langle s, t : \ tst^{-1} = s^{-1}, t^2 = e \rangle$. Consider two minimal (topologically free) actions of D_∞ on a compact Hausdorff infinite space. Then coe between them implies conjugacy.

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On the negative side, we have

Theorem (J. 2021)

There exists non-trivial finite groups F such that $G := \mathbb{Z} \times F$ admits two minimal free coe actions which are not conjugate.

Some useful notation and observation

Let D_∞ be the infinite dihedral group.

We think of $D_\infty = \mathbb{Z} \rtimes \frac{\mathbb{Z}}{2\mathbb{Z}} = \langle s \rangle \rtimes \langle t \rangle$ as an infinite two-leg ladder as follows ($\xrightarrow{g}$ denotes $\cdot g$):

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\cdots & & s^{-2}t & \xrightarrow{s^{-1}} & s^{-1}t & \xrightarrow{s^{-1}} & t & \xrightarrow{s^{-1}} & st & \xrightarrow{s^{-1}} & s^2t & \xrightarrow{s^{-1}} & s^3t & & \cdots \\
& & \uparrow t & & \uparrow t & & \uparrow t & & \uparrow t & & \uparrow t & & \uparrow t & & \\
\cdots & & s^{-2} & \xrightarrow{s} & s^{-1} & \xrightarrow{s} & e & \xrightarrow{s} & s & \xrightarrow{s} & s^2 & \xrightarrow{s} & s^3 & & \cdots
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\end{array}$$

Let $S = \{s^{\pm 1}, t\}$. D_∞ admits a right invariant word metric $d(g,h) := |hg^{-1}|_S$, where $|\cdot|_S$ denotes the word length, e.g. $|s^n t|_S = n+1$, $|s^n|_S = n$.

Proof of rigidity part

Goal: show the orbit cocycle $c : D_\infty \times X \to D_\infty$ satisfies that $c \sim \rho \in Aut(D_\infty)$.

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Step 1: show for each $x \in X$, either $\sup_{n\in\mathbb{Z}} d(c(s^n,x),s^n) < \infty$ or $\sup_{n\in\mathbb{Z}} d(c(s^n,x),s^{-n}) < \infty$.

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The bottom leg in the 1st ladder is mapped to the one (perhaps in reverse order) in the 2nd ladder up to shifting by a finite distance.

Step 2: Define

$$X_+ = \{x \in X : \sup_{n \in \mathbb{Z}} d(c(s^n, x), s^n) < \infty\}$$

$$X_- = \{x \in X : \sup_{n \in \mathbb{Z}} d(c(s^n, x), s^{-n}) < \infty\}.$$

Show that both X_+ and X_- are clopen subsets of X.

Step 3: Extract a cocycle with finite range, i.e. define

$$a : \mathbb{Z} \times X \to D_\infty$$
$$a(s^n, x) := \begin{cases} c(s^n, x)s^{-n}, & \text{if } x \in X_+ \\ c(s^n, x)s^n, & \text{if } x \in X_-. \end{cases}$$

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Step 4: Define $D : X \to D_\infty$ by

$$D(x) = \begin{cases} t & \text{if } x \in X_- \\ e & \text{if } x \in X_+. \end{cases}$$

Show that $a'(s^n, x) := D(s^n x)a(s^n, x)D(x)^{-1} \in \mathbb{Z}$, i.e.

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From above, we have

$$\begin{aligned} c &: D_\infty \times X \to D_\infty \\ \dashrightarrow a &: \mathbb{Z} \times X \to D_\infty \text{ (with finite range)} \\ \dashrightarrow a' &: \mathbb{Z} \times X \to \mathbb{Z} \text{ (with finite range).} \end{aligned}$$

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Step 5: Observe that the number of minimal $\mathbb{Z}$-components is either one or two (since $X = \overline{\mathbb{Z}x} \cup \overline{\mathbb{Z}tx}$ for all $x \in X$).

We split the proof into two cases:

From above, we have

$$\begin{aligned} &c : D_\infty \times X \to D_\infty \\ \dashrightarrow\ &a : \mathbb{Z} \times X \to D_\infty \text{ (with finite range)} \\ \dashrightarrow\ &a' : \mathbb{Z} \times X \to \mathbb{Z} \text{ (with finite range).} \end{aligned}$$

Step 5: Observe that the number of minimal $\mathbb{Z}$-components is either one or two (since $X = \overline{\mathbb{Z}x} \cup \overline{\mathbb{Z}tx}$ for all $x \in X$).

We split the proof into two cases:

Case I: $G \curvearrowright X$ has only one minimal $\mathbb{Z}$-component.

Case II: $G \curvearrowright X$ has two minimal $\mathbb{Z}$-components.

Case I: We use

Theorem (Gottschalk-Hedlund 1955)

For any minimal action $\mathbb{Z} \curvearrowright X$, every continuous cocycle $c : \mathbb{Z} \times X \to \mathbb{R}$ with finite range is a coboundary.

to deduce that a' is a coboundary, which implies $c : \mathbb{Z} \times X \to D_\infty$ is cohomologous to $id : \mathbb{Z} \to \mathbb{Z} \subset D_\infty$.

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Then use t is a reflection of $\mathbb{Z} = \langle s \rangle$ to deduce that $c \sim \rho \in Aut(D_\infty)$, where $\rho(s) = s$ and $\rho(t) = s^k t$ for some $k \in \mathbb{Z}$.

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Case II: a little more involved, but can be handled similarly as in Case I.

This finishes the proof of the rigidity part.

Proof of non-rigidity part

Goal: show that for certain finite group F, $F \times \mathbb{Z}$ admits two coe actions which are not conjugate.

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Idea: let c, c' be any two continuous cocycles $\mathbb{Z} \times X \to F$, where $\alpha : \mathbb{Z} \curvearrowright X$ is an action TBD.

Consider skew product actions $\widetilde{\alpha}$ and $\widetilde{\alpha'}$, where $\widetilde{\alpha}$ is defined as follows (and $\widetilde{\alpha'}$ is defined similarly)

$$\widetilde{\alpha} : \; F \times \mathbb{Z} \curvearrowright X \times F$$
$$\widetilde{\alpha}_{(f,n)}(x, f') = (\alpha_n(x), c(n,x)f'f^{-1}).$$

Observations:

- $\widetilde{\alpha} \sim_{coe} \widetilde{\alpha'}$.

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It suffices to construct an action $\mathbb{Z} \curvearrowright X$ with two cocycles c, c' such that $c \not\sim c'$.

Open questions

Here are some open questions related to coe.

1. Let $\mathcal{G} = \{G : \text{ coe} \Leftrightarrow \text{conjugacy for any two minimal actions of } G\}$. Study the membership problem for $\mathcal{G}$.
2. Is the full shift $F_n \curvearrowright A^{F_n}$ coe rigid action?
3. When is a generalized full shift $G \curvearrowright X_0^I$ coe rigid action?
4. Does every countably infinite group G admit continuum many pairwise non-coe actions?
5. Find more (computable) invariants to distinguish actions up to coe.

Thank you for your attention!