C*-algebras and Leavitt path algebras for labelled graphs

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Cuntz-Krieger algebras and generalisations

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Labelled graphs

- By a (directed) graph we mean a quadruple $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ where $\mathcal{E}^0, \mathcal{E}^1$ are sets, $s : \mathcal{E}^1 \to \mathcal{E}^0$ and $r : \mathcal{E}^1 \to \mathcal{E}^0$ are maps.
- Given a set $\mathcal{A}$, which is thought as a set of letters, an (edge-)labelling on a graph $\mathcal{E}$ is an onto map $\mathcal{L} : \mathcal{E}^1 \to \mathcal{A}$.
- We call the pair $(\mathcal{E}, \mathcal{L})$ a labelled graph.
- A path λ on $\mathcal{E}$ is a sequence (finite or infinite) of edges $\lambda = \lambda_1 \ldots \lambda_n(\ldots)$ such that $r(\lambda_i) = s(\lambda_{i+1})$ $\forall i$.
- We can extend the map $\mathcal{L}$ to any path λ by $\mathcal{L}(\lambda) = \mathcal{L}(\lambda_1) \ldots \mathcal{L}(\lambda_n)(\ldots)$.
- An element $\alpha = \mathcal{L}(\lambda)$ is called a labelled path. We also include the empty word ω as a labelled path.
- The set of finite labelled paths will be denoted by $\mathcal{L}^*$ and the set of infinite labelled paths by $\mathcal{L}^\infty$. We also define $\mathcal{L}^{\geq 1} = \mathcal{L}^* \setminus \{\omega\}$ and $\mathcal{L}^{\leq \infty} = \mathcal{L}^* \cup \mathcal{L}^\infty$.

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Labelled spaces

- For $\alpha \in \mathcal{L}^*$ and $A \in \mathcal{P}(\mathcal{E}^0)$, the relative range of α with respect to A is

$$r(A, \alpha) = \{r(\lambda) \mid \lambda \in \mathcal{E}^*,\ \mathcal{L}(\lambda) = \alpha,\ s(\lambda) \in A\}$$

if $\alpha \in \mathcal{L}^{\geq 1}$ and $r(A, \omega) = A$ if $\alpha = \omega$. We define $r(\alpha) := r(\mathcal{E}^0, \alpha)$.

Definition

A (normal weakly left-resolving) labelled space is a triple $(\mathcal{E}, \mathcal{L}, \mathcal{B})$, where $(\mathcal{E}, \mathcal{L})$ is a labelled graph and $\mathcal{B} \subseteq \mathcal{P}(\mathcal{E}^0)$ is a (not necessarily unital) Boolean algebra such that

- $r(\alpha) \in \mathcal{B}$ and $r(A, \alpha) \in \mathcal{B}$ for all $\alpha \in \mathcal{L}^{\geq 1}$ and $A \in \mathcal{B}$,
- $r(A \cap B, \alpha) = r(A, \alpha) \cap r(B, \alpha)$ for all $\alpha \in \mathcal{L}^{\geq 1}$ and $A, B \in \mathcal{B}$.

For each $A \in \mathcal{B}$, we let $\Delta_A = \{a \in \mathcal{A} \mid r(A, a) \neq \emptyset\}$. We say that $A \in \mathcal{B}$ is regular if for all $B \in \mathcal{B}$ such that $B \subseteq A$, we have that $0 < |\Delta_B| < \infty$. The set of regular sets is denoted by $\mathcal{B}_{reg}$.

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The C*-algebra of a labelled space

Definition (Bates, Pask - 2007, 2009; Bates, Carlsen, Pask - 2015)

Let $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ be labelled space. The C^*-algebra associated with $(\mathcal{E}, \mathcal{L}, \mathcal{B})$, denoted by $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B})$, is the universal C^*-algebra generated by projections $\{p_A \mid A \in \mathcal{B}\}$ and partial isometries $\{s_a \mid a \in \mathcal{A}\}$ subject to the relations

(i) $p_{A\cap B} = p_A p_B$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ and $p_\emptyset = 0$, for every $A, B \in \mathcal{B}$;

(ii) $p_A s_a = s_a p_{r(A,a)}$, for every $A \in \mathcal{B}$ and $a \in \mathcal{A}$;

(iii) $s_a^* s_a = p_{r(a)}$ and $s_b^* s_a = 0$ if $b \neq a$, for every $a, b \in \mathcal{A}$;

(iv) For every $A \in \mathcal{B}_{reg}$,

$$p_A = \sum_{a \in \Delta_A} s_a p_{r(A,a)} s_a^*.$$

Leavitt path algebras for labelled graphs

Definition

Let $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ be a labelled space and R a unital commutative ring. The Leavitt labelled path algebra associated with $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ with coefficients in R, denoted by $L_R(\mathcal{E}, \mathcal{L}, \mathcal{B})$, is the universal R-algebra with generators $\{p_A \mid A \in \mathcal{B}\}$ and $\{s_a, s_a^* \mid a \in \mathcal{A}\}$ subject to the relations

(i) $p_{A\cap B} = p_A p_B$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ and $p_\emptyset = 0$, for every $A, B \in \mathcal{B}$;

(ii) $p_A s_a = s_a p_{r(A,a)}$ and $s_a^* p_A = p_{r(A,a)} s_a^*$, for every $A \in \mathcal{B}$ and $a \in \mathcal{A}$;

(iii) $s_a^* s_a = p_{r(a)}$ and $s_b^* s_a = 0$ if $b \neq a$, for every $a, b \in \mathcal{A}$;

(iv) $s_a s_a^* s_a = s_a$ and $s_a^* s_a s_a^* = s_a^*$ for every $a \in \mathcal{A}$;

(v) For every $A \in \mathcal{B}_{reg}$,

$$p_A = \sum_{a \in \mathcal{L}(A\mathcal{E}^1)} s_a p_{r(A,a)} s_a^*.$$

Example - graph algebras

Proposition (Bates, Pask - 2007, BdCGvW - 2021*)

Let $\mathcal{E}$ be a graph and consider $\mathcal{A} = \mathcal{E}^1$, $\mathcal{L} = Id_{\mathcal{E}^1}$ and $\mathcal{B} = \{A \subseteq \mathcal{E}^0 \mid A \text{ is finite}\}$. Then $C^*(\mathcal{E}) \cong C^*(\mathcal{E}, \mathcal{L}, \mathcal{B})$ and $L_R(\mathcal{E}) \cong L_R(\mathcal{E}, \mathcal{L}, \mathcal{B})$.

Definition

Let $(\mathcal{E}, \mathcal{L})$ be a labelled graph. We say that $(\mathcal{E}, \mathcal{L})$ is left-resolving if for every $v \in \mathcal{E}^0$, we have that $\mathcal{L}|_{r^{-1}(v)}$ is injective. We say that $(\mathcal{E}, \mathcal{L})$ is label-finite if $|\mathcal{L}^{-1}(a)| < \infty$ for every $a \in \mathcal{A}$.

Proposition (Bates, Pask - 2007, BdCGvW - 2021*)

Let $(\mathcal{E}, \mathcal{L})$ be a left-resolving, label-finite, labelled graph, and let $\mathcal{B}$ be the family of all finite subsets of $\mathcal{E}^0$. Then $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong C^*(\mathcal{E})$ and $L_R(\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong L_R(\mathcal{E})$.

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Example - commutative algebras

Let $\mathcal{B}$ be a Boolean algebra. A filter in $\mathcal{B}$ is a set $\mathcal{F} \subseteq \mathcal{B}$ such that $A \cap B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$ and whenever $B \in \mathcal{B}$ is such that $A \subseteq B$ for some $A \in \mathcal{F}$, we have that $B \in \mathcal{F}$. An ultrafilter in $\mathcal{B}$ is a proper maximal filter.

The Stone dual of $\mathcal{B}$ is the set X of all ultrafilters with a basis of compact-open sets given by sets of the form $U_A = \{\mathcal{F} \in X \mid A \in \mathcal{F}\}$ for $A \in \mathcal{B}$. In fact, all compact-open sets of X are of the form U_A for some A and the map $A \mapsto U_A$ is a Boolean algebra isomorphism.

Proposition

Let $\mathcal{B}$ be a Boolean algebra and let X be its Stone dual. Define $\mathcal{E}^0 = X$, $\mathcal{A} = \mathcal{E}^1 = \emptyset$ and $\mathcal{L}$ the empty function. Then $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) = C_0(X)$ and $L_R(\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong C_C(X, R)$.

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Theorem (Keimel - 1970)

Suppose that R is an integral domain and let A be a torsion-free commutative algebra generated by idempotents. Let $\mathcal{B}$ be the Boolean algebra of idempotents of A. Then $A \cong C_C(X, R)$, where X is the Stone dual of $\mathcal{B}$.

Example

Let $\overline{\mathbb{N}}$ be the one-point compactification of $\mathbb{N}$ and let $\mathcal{B}$ be the Boolean algebra of compact-open sets of $\overline{\mathbb{N}}$. Suppose that R is an integral domain. Then $C_C(\overline{\mathbb{N}}, R)$ is a Leavitt labelled path algebra that is not a Leavitt path algebra for any graph or ultragraph.

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Groupoid model

Fix $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ a labelled space. For each $\alpha \in \mathcal{L}^*$, we have that $\mathcal{B}_\alpha = \{A \in \mathcal{B} \mid A \subseteq r(\alpha)\}$ is a Boolean algebra, which is unital whenever $\alpha \neq \omega$. We denote by X_α the corresponding Stone dual.

For each $a \in \mathcal{A}$, we define two maps $f_a : X_a \to X_\omega \cup \{\emptyset\}$ and $h_a : X_a \to X_\omega$ by

$$f_a(\mathcal{F}) = \{A \in \mathcal{B} \mid r(A, a) \in \mathcal{F}\}$$

and

$$h_a(\mathcal{F}) = \{A \in \mathcal{B} \mid \exists B \in \mathcal{F} \text{ s.t. } B \subseteq A\}$$

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Let $E^0 = X_\omega$, $F^0 = X_\omega \cup \{\emptyset\}$ its one-point extension and $E^1 = \bigsqcup_{a\in\mathcal{A}} X_a$ with the disjoint topology. We denote an element of E^1 by $e^a_{\mathcal{F}}$ for $a \in \mathcal{A}$ and $\mathcal{F} \in X_a$. Define maps $s : E^1 \to F^0$ by $s(e^a_{\mathcal{F}}) = f_a(\mathcal{F})$ and $r : E^1 \to E^0$ by $r(e^a_{\mathcal{F}}) = h_a(\mathcal{F})$. Then (E^1, s, r) is a topological correspondence from F^1 to E^1.

Because $E^0 \subseteq F^0$, we can define the set of finite paths E^* and infinite paths E^∞ in the usual way. Let $E^0_{reg} = \{v \in E^0 \mid \exists V \text{ cpt. ngbh. of } v \text{ s.t. } s^{-1}(V) \text{ is cpt. and } V = s(s^{-1}(V))\}$. The boundary path space is the set $\partial E = \{\mu \in E^* \mid r(\mu) \notin E^0_{reg}\} \cup E^\infty$.

We can then consider the boundary path groupoid

$$\Gamma(E) = \{(\mu\gamma, |\mu| - |\nu|, \nu\gamma) \in \partial E \times \mathbb{Z} \times \partial E \mid \mu, \nu \in E^*, \gamma \in \partial E\},$$

which is an amenable étale ample Hausdorff groupoid with an appropriate topology.

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Theorem (de C., Kang - 2021*)

The boundary path groupoid is isomorphic as topological groupoids to the tight groupoid of a certain inverse semigroup.

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Let $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ be a labelled space. Then $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong C^*(\Gamma(E))$.

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Effectiveness

Definition

Let $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ be a labelled space.

(i) A pair (α, A) with $\alpha \in \mathcal{L}^{\geq 1}$ and $A \in \mathcal{B}_\alpha$ is a cycle if for every $B \in \mathcal{B}_\alpha$ with $B \subseteq A$, we have that $r(B, \alpha) = B$.

(ii) A cycle (α, A) has an exit if there exist $0 \leq k \leq |\alpha|$ and $\emptyset \neq B \in \mathcal{B}$ such that $B \subseteq r(A, \alpha_{1,k})$ and $\Delta_B \neq \{\alpha_{k+1}\}$ (where $\alpha_{|\alpha|+1} = \alpha_1$).

(iii) The labelled space $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ satisfies condition (L) if every cycle has an exit.

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Minimality

Definition

Let $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ be a normal labelled space. A subset H of $\mathcal{B}$ is hereditary if the following conditions hold:

(i) $r(A, \alpha) \in H$ for all $A \in H$ and all $\alpha \in \mathcal{L}^*$,

(ii) $A \cup B \in H$ for all $A, B \in H$,

(iii) if $B \in \mathcal{B}$ is such that $B \subseteq A$ for some $A \in H$, then $B \in H$.

A hereditary set H is saturated if given $A \in \mathcal{B}_{reg}$ such that $r(A, a) \in H$ for all $a \in \mathcal{A}$, then $A \in H$.

Proposition (COP 2017, dCvW 2020)

The groupoid $\Gamma(E)$ is minimal if and only if $\{\emptyset\}$ and $\mathcal{B}$ are the only hereditary saturated subsets of $\mathcal{B}$.

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Simplicity

Theorem (COP - 2017, dCvW - 2020, BdCGvW - 2021*)

Let $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ *be a labelled space and R a field. The following are equivalent:*

(i) $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B})$ *is simple;*

(ii) $L_R(\mathcal{E}, \mathcal{L}, \mathcal{B})$ *is simple;*

(iii) $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ *satisfies condition (L) and* $\{\emptyset\}$ *and* $\mathcal{B}$ *are the only hereditary saturated subsets of* $\mathcal{B}$*.*

Remark

The condition that R is a field is necessary for $L_R(\mathcal{E}, \mathcal{L}, \mathcal{B})$ to be simple.

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Partial actions and labelled spaces

Definition

A partial action of a group G on a topological space X is a pair $\Phi = (\{U_t\}_{t\in G}, \{\phi_t\}_{t\in G})$ consisting of a collection $\{U_t\}_{t\in G}$ of open subsets of X and a collection $\{\phi_t\}_{t\in G}$ of homeomorphisms, $\phi_t : U_{t^{-1}} \to U_t$, such that

(i) $U_e = X$ and ϕ_e is the identity on X,

(ii) $\phi_s(U_{s^{-1}} \cap U_t) = U_s \cap U_{st}$,

(iii) $\phi_s(\phi_t(x)) = \phi_{st}(x)$ for every $x \in U_{t^{-1}} \cap U_{(st)^{-1}}$.

If the partial action is given by the free group $\mathbb{F}$ on a set of generators, then the partial action is semi-saturated if

$$\phi_s \circ \phi_t = \phi_{st}$$

for every $s, t \in \mathbb{F}$ such that $|st| = |s| + |t|$, and orthogonal if $U_a \cap U_b = \emptyset$ for a, b in the set of generator with $a \neq b$.

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Suppose that G is a discrete group and X is a Hausdorff space with a basis of compact-set (ie, a Stone space). Given a partial action Φ of G on X, we have dual actions $\hat{\Phi}$ on $C_0(X)$ and $C_C(X, R)$. From this, we can build:

- a transformation groupoid $G \ltimes_\Phi X$;
- a C*-algebra $C_0(X) \rtimes_{\hat{\Phi}} G$;
- an R-algebra $C_C(X, R) \rtimes_{\hat{\Phi}} G$.

Theorem (Abadie - 2003)

$C^*(G \ltimes_\Phi X) \cong C_0(X) \rtimes_{\hat{\Phi}} G.$

Theorem (Beuter, Gonçalves - 2018)

$A_R(G \ltimes_\Phi X) \cong C_C(X, R) \rtimes_{\hat{\Phi}} G.$

Suppose that G is a discrete group and X is a Hausdorff space with a basis of compact-set (ie, a Stone space). Given a partial action Φ of G on X, we have dual actions $\hat{\Phi}$ on $C_0(X)$ and $C_C(X, R)$. From this, we can build:

- a transformation groupoid $G \ltimes_\Phi X$;
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Theorem (d C., van Wyk - 2020)

There is an orthogonal semi-saturated partial action Φ *of* $\mathbb{F}$ *on* ∂E *such that* $\Gamma(E) \cong \mathbb{F} \ltimes_{\Phi} \partial E$.

Corollary

- $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong C_0(\partial E) \rtimes_{\hat{\Phi}} \mathbb{F}$.
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Theorem (BdCGvW - 2021*)

Let X be a Stone space and let $\rho = (\{U_t\}_{t\in\mathbb{F}}, \{\rho_t\}_{t\in\mathbb{F}})$ be a semi-saturated, orthogonal topological partial action of a free group $\mathbb{F}$ on X such that U_t is compact-open for all $t \in \mathbb{F} \setminus \{\omega\}$. Then there exists a labelled space $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ and a homeomorphism $f : X \to \partial E$, where ∂E, such that f is equivariant with respect to the actions ρ and Φ given by the above theorem. In particular $\mathbb{F} \ltimes_\rho X$ and $\mathbb{F} \ltimes_\varphi \partial E$ are isomorphic as topological groupoids.

Corollary (BdCGvW - 2021*)

Let R be an integral domain and A be a torsion-free commutative R-algebra generated by its idempotents elements. Let also $\tau = (\{D_t\}_{t\in\mathbb{F}}, \{\tau_t\}_{t\in\mathbb{F}})$ be a semi-saturated, orthogonal algebraic partial action of a free group $\mathbb{F}$ on A such that D_t is unital for every $t \in \mathbb{F} \setminus \{\omega\}$. Then, there exists a labelled space $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ such that $A \rtimes_\tau \mathbb{F} \cong L_R(\mathcal{E}, \mathcal{L}, \mathcal{B})$.

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Thank you!