Classifying sufficiently connected PSC manifolds in dimensions 4 and 5

Chao Li
(Joint with Otis Chodosh and Yevgeny Liokumovich)

Dec 3, 2021
The Global Noncommutative Geometry Seminar

Scalar curvature

Scalar curvature R is a weak curvature invariant one can attach pointwise to a Riemannian n-manifold. Its value at every point can be described in several different ways:

Scalar curvature

Scalar curvature R is a weak curvature invariant one can attach pointwise to a Riemannian n-manifold. Its value at every point can be described in several different ways:

1. R is the trace of the Ricci tensor;
2. R is twice the sum of sectional curvatures over all 2-planes.

Scalar curvature

Scalar curvature R is a weak curvature invariant one can attach pointwise to a Riemannian n-manifold. Its value at every point can be described in several different ways:

1. R is the trace of the Ricci tensor;
2. R is twice the sum of sectional curvatures over all 2-planes.
3. R determines the volume of a small geodesic ball:

$$V(B_r(p)) = \omega_n r^n \left(1 - \frac{R(p)}{6(n+2)} r^2 + \cdots \right).$$

Thus, $R > 0 \Rightarrow$ small geodesic balls have smaller volume than Euclidean.

The Kazdan-Warner theorem

Kazdan-Warner, 1975: Let $n \geq 3$. Any $f(x) \in C^\infty(M)$ with $f(x) < 0$ for some $x \in M$ can be realized as the scalar curvature function of some Riemannian metric on M.

The Kazdan-Warner theorem

Kazdan-Warner, 1975: Let $n \geq 3$. Any $f(x) \in C^\infty(M)$ with $f(x) < 0$ for some $x \in M$ can be realized as the scalar curvature function of some Riemannian metric on M.

On the other hand, there are known topological obstructions to existence of positive scalar curvature metrics. The first such obstruction is the vanishing of the $\hat{A}$ genus (a certain characteristic number).

The Kazdan-Warner theorem

Kazdan-Warner, 1975: Let $n \geq 3$. Any $f(x) \in C^{\infty}(M)$ with $f(x) < 0$ for some $x \in M$ can be realized as the scalar curvature function of some Riemannian metric on M.

On the other hand, there are known topological obstructions to existence of positive scalar curvature metrics. The first such obstruction is the vanishing of the $\hat{A}$ genus (a certain characteristic number).

Example

K3 surfaces. A 4-dimensional manifold, $\hat{A} = 2 \Rightarrow$ *no PSC metric. On the other hand, it has a metric with* $\mathrm{Ric} = 0$ *(in particular* $R = 0$*), by the Calabi-Yau theorem.*

The obstruction problem

Question (The obstruction problem)

Which smooth, closed manifolds M^n admit a Riemannian metric with positive scalar curvature?

Long history, lots of exciting results!

Low dimensions

In 2D, $R = 2K$. Complete classification by the Gauss-Bonnet theorem:

$$M^2 \text{ admits a PSC metric } \Leftrightarrow M^2 \text{ is diffeomorphic to } S^2 \text{ or } \mathbb{R}P^2.$$

Low dimensions

In 2D, $R = 2K$. Complete classification by the Gauss-Bonnet theorem:

$$M^2 \text{ admits a PSC metric } \Leftrightarrow M^2 \text{ is diffeomorphic to } S^2 \text{ or } \mathbb{R}P^2.$$

In 3D, 3-manifold topology (including the Poincaré conjecture) $\Rightarrow$ each closed orientable M^3 decomposes as a connected sum:

$$M = (S^3/\Gamma_1)\# \cdots \#(S^3/\Gamma_p)\# q \cdot (S^2 \times S^1)\# K_1 \# \cdots \# K_m.$$

Here each Γ_j is a finite group acting freely on S^3, and each K_j is an *asphericial* 3-manifold (each K_j is covered by $\mathbb{R}^3$).

Recall: a closed smooth manifold is aspherical, if its universal cover is contractible.

Recall: a closed smooth manifold is aspherical, if its universal cover is contractible.

Examples:

- All 2D surfaces of genus ≥ 1;
- All manifolds admitting a metric of sec ≤ 0 (e.g. the torus and all hyperbolic manifolds).

Recall: a closed smooth manifold is aspherical, if its universal cover is contractible.

Examples:

- All 2D surfaces of genus ≥ 1;
- All manifolds admitting a metric of sec ≤ 0 (e.g. the torus and all hyperbolic manifolds).

Theorem (Schoen-Yau, Gromov-Lawson, $\sim$ 1980)

A closed orientable 3-manifold M admits a PSC metric $\Leftrightarrow$ there is no aspherical component in the decomposition. ($\Leftrightarrow$ a finite cover is diffeomorphic to S^3 or connected sums of $S^2 \times S^1$.)

Recall: a closed smooth manifold is asphericall, if its universal cover is contractible.

Examples:

- All 2D surfaces of genus ≥ 1;
- All manifolds admitting a metric of sec ≤ 0 (e.g. the torus and all hyperbolic manifolds).

Theorem (Schoen-Yau, Gromov-Lawson, $\sim$ 1980)

A closed orientable 3-manifold M admits a PSC metric $\Leftrightarrow$ there is no aspherical component in the decomposition. ($\Leftrightarrow$ a finite cover is diffeomorphic to S^3 or connected sums of $S^2 \times S^1$.)

In particular:

Corollary

A closed aspherical 3-manifold does not admit a PSC metric.

In dimensions $n \geq 4$

A few landmarks.

- Lichnerowicz, 1963: M^{4k} is spin and has PSC metric $\Rightarrow \hat{A} = 0$. Greatly expanded by Hitchin, 1974.
- Schoen-Yau, Gromov-Lawson, $\sim$ 1980: the n-torus T^n admits no PSC metric. ($\hat{A}(T^n) = 0$.)

Remark: The theorem also holds for $T^n \# M^n$, M is closed. This implies (via the Lohkamp's reduction) the positive mass theorem for asymptotically flat manifolds in dimension n.

In dimensions $n \geq 4$

A few landmarks.

- Lichnerowicz, 1963: M^{4k} is spin and has PSC metric $\Rightarrow \hat{A} = 0$. Greatly expanded by Hitchin, 1974.
- Schoen-Yau, Gromov-Lawson, $\sim$ 1980: the n-torus T^n admits no PSC metric. ($\hat{A}(T^n) = 0$.)

Remark: The theorem also holds for $T^n \# M^n$, M is closed. This implies (via the Lohkamp's reduction) the positive mass theorem for asymptotically flat manifolds in dimension n.

Chodosh-L : $T^n \# M^n$ does not have any complete PSC metric, M may be noncompact.

Lesourd-Unger-Yau: Positive mass theorem on asymptotically Schwarszchild manifolds for arbitrary ends.

In dimensions $n \geq 4$

- Gromov-Lawson, 1980, and Stolz, 1992: characterization of simply connected manifolds of dimensions at least 5.
- Witten, 1994: In dimension 4, Seiberg-Witten invariants obstruct PSC metrics.

In dimensions $n \geq 4$

- Gromov-Lawson, 1980, and Stolz, 1992: characterization of simply connected manifolds of dimensions at least 5.
- Witten, 1994: In dimension 4, Seiberg-Witten invariants obstruct PSC metrics.

Conjecture A (Schoen-Yau, 1987)

A closed aspherical manifold does not admit any Riemannian metric of positive scalar curvature.

Large scale geometry

A more ambitious formulation in metric geometry (Large Riemannian manifolds, Gromov, 1986):

Conjecture B

Suppose M^n is a "large" Riemannian manifold. Then for every $R > 0$

$$\sup_{p \in M} V(B_R(p)) \geq \omega_n R^n,$$

here ω_n is the volume of the unit ball in R^n.

Large scale geometry

A more ambitious formulation in metric geometry (Large Riemannian manifolds, Gromov, 1986):

Conjecture B

Suppose M^n is a "large" Riemannian manifold. Then for every $R > 0$

$$\sup_{p \in M} V(B_R(p)) \geq \omega_n R^n,$$

here ω_n is the volume of the unit ball in $\mathbb{R}^n$.

"Large" Riemannian manifolds include complete contractible Riemannian manifolds M with an isometry group G acting cocompactly on M.

Large scale geometry

A more ambitious formulation in metric geometry (Large Riemannian manifolds, Gromov, 1986):

Conjecture B

Suppose M^n is a "large" Riemannian manifold. Then for every $R > 0$

$$\sup_{p \in M} V(B_R(p)) \geq \omega_n R^n,$$

here ω_n is the volume of the unit ball in R^n.

"Large" Riemannian manifolds include complete contractible Riemannian manifolds M with an isometry group G acting cocompactly on M.

Thus, the universal cover of a closed aspherical Riemannian manifold is large. Conjecture B $\Rightarrow$ Conjecture A.

Relations with metric/topological properties

Rosenberg (1983): If a form of (still unproven) strong Novikov conjecture holds, then Conjecture A holds.

Relations with metric/topological properties

Rosenberg (1983): If a form of (still unproven) strong Novikov conjecture holds, then Conjecture A holds.

Gromov pointed out connections between Conjecture A and metric properties of Riemannian manifolds with $R > 1$, including the Urysohn width, the waist and the filling radius of Riemannian manifolds with scalar curvature lower bounds.

Relations with metric/topological properties

Rosenberg (1983): If a form of (still unproven) strong Novikov conjecture holds, then Conjecture A holds.

Gromov pointed out connections between Conjecture A and metric properties of Riemannian manifolds with $R > 1$, including the Urysohn width, the waist and the filling radius of Riemannian manifolds with scalar curvature lower bounds.

Conjecture C

Suppose (M^n, g) is complete with $R(g) \geq 1$. Then there is a constant $C(n)$, an $(n-2)$-dimensional simplicial complex, and a continuous map $\varphi : M \to P$, such that for any $p \in P$, $\operatorname{diam}_g(\varphi^{-1}(p)) \leq C(n)$.

Some previous progress

- When $n = 4$, in a 1987 survey paper, Schoen-Yau proposed an outline of the conjecture, but certain key parts are missing. This outline is important to us.
- Greene - Petersen, 1992: A local, very coarse version of Conjecture B.
- Yu, 1999: Conjecture A holds, if universal cover of M has sub-exponential volume growth.
- Dranishnikov, 2002: A further extension of Yu's work.
- Guth, 2011: A coarse version of Conjecture B holds: for all $R > 0$,
$$\sup_{p \in M} Vol(B_R(p)) \geq \delta(n)\omega_n R^n$$
- J. Wang, 2019: Conjecture A holds for four dimensional aspherical manifolds with $b_1 > 0$.
- Liokumovich-Maximo: Conjecture C holds in dimension 3.

Main theorems

Theorem 1 (Chodosh-L, 2020)

A closed aspherical manifold of dimension 4 *or* 5 *does not admit a Riemannian metric with positive scalar curvature; any metric with* $R \geq 0$ *is flat.*

Main theorems

Theorem 1 (Chodosh-L, 2020)

A closed aspherical manifold of dimension 4 *or* 5 *does not admit a Riemannian metric with positive scalar curvature; any metric with* $R \geq 0$ *is flat.*

Remark: Gromov proved the result in 5 dimensions independently.

Main theorems

Theorem 1 (Chodosh-L, 2020)

A closed aspherical manifold of dimension 4 *or* 5 *does not admit a Riemannian metric with positive scalar curvature; any metric with* $R \geq 0$ *is flat.*

Remark: Gromov proved the result in 5 dimensions independently.

There is a more general mapping version of the theorem.

Theorem 2 (Chodosh-L-Liokumovich, 2021)

Let (N^n, g) be a closed manifold with positive scalar curvature, $n \in \{4, 5\}$. We assume:

- $\pi_2(N) = 0$, *if* $n = 4$;
- $\pi_2(N) = \pi_3(N) = 0$, *if* $n = 5$.

Then a finite cover $\hat{N}$ of N is homotopy equivalent to S^n or connected sums of $S^{n-1} \times S^1$.

Theorem 2 (Chodosh-L-Liokumovich, 2021)

Let (N^n, g) be a closed manifold with positive scalar curvature, $n \in \{4, 5\}$. We assume:

- $\pi_2(N) = 0$, *if* $n = 4$;
- $\pi_2(N) = \pi_3(N) = 0$, *if* $n = 5$.

Then a finite cover $\hat{N}$ of N is homotopy equivalent to S^n or connected sums of $S^{n-1} \times S^1$.

Remarks:

1. If $n = 4$ and $\hat{N}$ is homotopy equivalent to S^4 or $S^3 \times S^1$, or if $n = 5$, then homotopy equivalence can be upgraded to homeomorphism. Diffeomorphism needs a more delicate approach.

Theorem 2 (Chodosh-L-Liokumovich, 2021)

Let (N^n, g) be a closed manifold with positive scalar curvature, $n \in \{4,5\}$. We assume:

- $\pi_2(N) = 0$, *if* $n = 4$;
- $\pi_2(N) = \pi_3(N) = 0$, *if* $n = 5$.

Then a finite cover $\hat{N}$ of N is homotopy equivalent to S^n or connected sums of $S^{n-1} \times S^1$.

Remarks:

1. If $n = 4$ and $\hat{N}$ is homotopy equivalent to S^4 or $S^3 \times S^1$, or if $n = 5$, then homotopy equivalence can be upgraded to homeomorphism. Diffeomorphism needs a more delicate approach.
2. The conditions on higher homotopy groups are necessary: consider $T^k \times S^{n-k}$.

Theorem 2 (Chodosh-L-Liokumovich, 2021)

Let (N^n, g) be a closed manifold with positive scalar curvature, $n \in \{4, 5\}$. We assume:

- $\pi_2(N) = 0$, *if* $n = 4$;
- $\pi_2(N) = \pi_3(N) = 0$, *if* $n = 5$.

Then a finite cover $\hat{N}$ of N is homotopy equivalent to S^n or connected sums of $S^{n-1} \times S^1$.

Remarks:

1. If $n = 4$ and $\hat{N}$ is homotopy equivalent to S^4 or $S^3 \times S^1$, or if $n = 5$, then homotopy equivalence can be upgraded to homeomorphism. Diffeomorphism needs a more delicate approach.
2. The conditions on higher homotopy groups are necessary: consider $T^k \times S^{n-k}$.
3. Heuristically, higher connectivity implies lower Urysohn dimension.

Connections to PIC

Conjecture (Gromov, Schoen)

Let (M^n, g) *be a closed manifold with positive isotropic curvature. Then* $\pi_1(M)$ *is virtually free, and in fact a finite cover of* M *is diffeomorphic to the connected sums of* S^n *and* $S^{n-1} \times S^1$*.*

Connections to PIC

Conjecture (Gromov, Schoen)

Let (M^n, g) be a closed manifold with positive isotropic curvature. Then $\pi_1(M)$ is virtually free, and in fact a finite cover of M is diffeomorphic to the connected sums of S^n and $S^{n-1} \times S^1$.

Approaches include:

- Minimal surfaces (Micallef-Moore, Fraser, etc.). $\pi_1(M) = \cdots = \pi_{[\frac{n}{2}]}(M) = 0$.
- Ricci flow (Brendle-Schoen, Brendle, Chen-Tang-Zhu, Huang, etc.). Conjecture holds in dimension 3 (Chen-Tang-Zhu), and in dimensions at least 12 (Brendle, Huang).

Connections to PIC

Conjecture (Gromov, Schoen)

Let (M^n, g) be a closed manifold with positive isotropic curvature. Then $\pi_1(M)$ is virtually free, and in fact a finite cover of M is diffeomorphic to the connected sums of S^n and $S^{n-1} \times S^1$.

Approaches include:

- Minimal surfaces (Micallef-Moore, Fraser, etc.). $\pi_1(M) = \cdots = \pi_{[\frac{n}{2}]}(M) = 0$.
- Ricci flow (Brendle-Schoen, Brendle, Chen-Tang-Zhu, Huang, etc.). Conjecture holds in dimension 3 (Chen-Tang-Zhu), and in dimensions at least 12 (Brendle, Huang).

Note: PIC implies PSC.

A more general mapping version

Theorem

Let $n \in \{4, 5\}$, (X^n, g) be a closed orientable manifold with positive scalar curvature, N^n be closed and orientable, $f : X \to N$ a map of nonzero degree. Suppose:

- $\pi_2(N) = 0$, *if* $n = 4$;
- $\pi_2(N) = \pi_3(N) = 0$, *if* $n = 5$.

Then a finite cover $\hat{N}$ of N is homotopy equivalent to S^n or connected sums of $S^{n-1} \times S^1$.

A more general mapping version

Theorem

Let $n \in \{4, 5\}$, (X^n, g) be a closed orientable manifold with positive scalar curvature, N^n be closed and orientable, $f : X \to N$ a map of nonzero degree. Suppose:

- $\pi_2(N) = 0$, *if* $n = 4$;
- $\pi_2(N) = \pi_3(N) = 0$, *if* $n = 5$.

Then a finite cover $\hat{N}$ of N is homotopy equivalent to S^n or connected sums of $S^{n-1} \times S^1$.

Corollary

Let $n \in \{4, 5\}$. If (X^n, g) is closed, orientable, and admits a nonzero degree map to an n-dimensional closed orientable $K(\pi, 1)$, then X does not admit any PSC metric.

Lemma

Let (N^n, g) be a closed Riemannian manifold, and $(\tilde{N}, \tilde{g})$ be its universal cover. Suppose $\tilde{N}$ is non-compact and $H_{n-1}(\tilde{N}, Z) = 0$. Then there exists a geodesic line $\sigma \subset \tilde{N}$ such that for any $L > 0$, there is a compact two-sided hypersurface with boundary $\hat{M}^{n-1} \subset \tilde{N}$ such that $d_{\tilde{g}}(\partial \hat{M}, \sigma) \geq 3L$ and $\hat{M}$ has nonzero algebraic intersection with σ.

Remark: the universal of a closed aspherical manifold is non-compact.

Key step: homological filling estimate

Theorem (Schoen-Yau)

Let (Σ^2, g) *be a 2D surface with* $\lambda_1(-\Delta_\Sigma + K_\Sigma) \geq \kappa/2 > 0$*.*
Then:

1. *If* $\partial\Sigma = \emptyset$ *then* $\operatorname{diam}\Sigma \leq 2\pi/\sqrt{\kappa}$*.*
2. *If* $\partial\Sigma \neq \emptyset$ *then* $d_g(p, \partial\Sigma) \leq 2\pi/\sqrt{\kappa}$ *for all* $p \in M$*.*

Key step: homological filling estimate

Theorem (Schoen-Yau)

Let (Σ^2, g) be a 2D surface with $\lambda_1(-\Delta_\Sigma + K_\Sigma) \geq \kappa/2 > 0$. Then:

1. *If $\partial\Sigma = \emptyset$ then $\operatorname{diam}\Sigma \leq 2\pi/\sqrt{\kappa}$.*
2. *If $\partial\Sigma \neq \emptyset$ then $d_g(p, \partial\Sigma) \leq 2\pi/\sqrt{\kappa}$ for all $p \in M$.*

Consequence:

Proposition

If γ is a null homologous curve in (M^3, g) with $R(g) \geq 1$, then $\gamma = \partial\Sigma$ for some Σ in $N_{2\pi}(\gamma)$.

Key step: homological filling estimate

Theorem (Schoen-Yau)

Let (Σ^2, g) *be a 2D surface with* $\lambda_1(-\Delta_\Sigma + K_\Sigma) \geq \kappa/2 > 0$.
Then:

1. *If* $\partial\Sigma = \emptyset$ *then* $\operatorname{diam}\Sigma \leq 2\pi/\sqrt{\kappa}$.
2. *If* $\partial\Sigma \neq \emptyset$ *then* $d_g(p, \partial\Sigma) \leq 2\pi/\sqrt{\kappa}$ *for all* $p \in M$.

Consequence:

Proposition

If γ *is a null homologous curve in* (M^3, g) *with* $R(g) \geq 1$, *then* $\gamma = \partial\Sigma$ *for some* Σ *in* $N_{2\pi}(\gamma)$.

Reason: the Plateau solution Σ with boundary γ has stability:

$$\int_\Sigma |\nabla f^2| - \tfrac{1}{2}(R_g - 2K_\Sigma + |A|^2)f^2 \geq 0, \quad \forall f \in C_0^1(\Sigma).$$

$\Rightarrow \lambda_1(-\Delta + K) \geq \frac{1}{2}$.

3D aspherical theorem

We prove a closed aspherical 3-manifold (N, g) does not admit a PSC metric. The proof is different from the original proofs of Schoen-Yau and Gromov-Lawson.

3D aspherical theorem

We prove a closed aspherical 3-manifold (N, g) does not admit a PSC metric. The proof is different from the original proofs of Schoen-Yau and Gromov-Lawson.

Proof:

Suppose the contrary, that (N^3, g) has $R(g) \geq 1$. Take its universal cover $(\tilde{N}, \tilde{g})$. Take σ and M with $L = 6\pi$. Then σ and M algebraically intersect nonzero.
However, since ∂M is null homologous, $\partial M = \partial \Sigma$ for $\Sigma \subset N_{2\pi}(\partial M)$. Thus $M \cup \Sigma$ is a 2-cycle intersecting σ nonzero algebraically, contradicting $H_2(\tilde{N}) = 0$. □

Take (N^n, g) a closed manifold as in Theorem 2. By the Hurewicz theorem, $\tilde{N}$ satisfies

$$H_1(\tilde{N}, \mathbb{Z}) = \cdots = H_{n-2}(\tilde{N}, \mathbb{Z}) = 0.$$

Take (N^n, g) a closed manifold as in Theorem 2. By the Hurewicz theorem, $\tilde{N}$ satisfies

$$H_1(\tilde{N}, \mathbb{Z}) = \cdots = H_{n-2}(\tilde{N}, \mathbb{Z}) = 0.$$

The key estimate is the following.

Theorem

Let $n \in \{4, 5\}$. There exists $L = L(N, g) > 0$, such that for any integral $(n-2)$-cycle Σ_{n-2} in $\tilde{N}$, there exists an $(n-1)$-chain Σ_{n-1} such that $\Sigma_{n-2} = \partial\Sigma_{n-1}$, and Σ_{n-1} is contained in the L-tubular neighborhood of Σ_{n-2}.

Take (N^n, g) a closed manifold as in Theorem 2. By the Hurewicz theorem, $\tilde{N}$ satisfies

$$H_1(\tilde{N}, \mathbb{Z}) = \cdots = H_{n-2}(\tilde{N}, \mathbb{Z}) = 0.$$

The key estimate is the following.

Theorem

Let $n \in \{4, 5\}$. There exists $L = L(N, g) > 0$, such that for any integral $(n-2)$-cycle Σ_{n-2} in $\tilde{N}$, there exists an $(n-1)$-chain Σ_{n-1} such that $\Sigma_{n-2} = \partial \Sigma_{n-1}$, and Σ_{n-1} is contained in the L-tubular neighborhood of Σ_{n-2}.

Combining this estimate and the lemma, we can prove Theorem 1 as before.

To prove Theorem 2, we start with the following observation, inspired by a work of Alpert-Balitsky-Guth.

Proposition

Assume that (N^n, g) has the property that any integral $(n-2)$-cycle in the universal cover can be filled in its L-tubular neighborhood. Then $(\tilde{N}, \tilde{g})$ satisfies:

- *For any $p \in \tilde{N}$, each connected component of a level set of $d(p, \cdot)$ has diameter $\leq 20L$.*

To prove Theorem 2, we start with the following observation, inspired by a work of Alpert-Balitsky-Guth.

Proposition

Assume that (N^n, g) has the property that any integral $(n-2)$-cycle in the universal cover can be filled in its L-tubular neighborhood. Then $(\tilde{N}, \tilde{g})$ satisfies:

- *For any $p \in \tilde{N}$, each connected component of a level set of $d(p, \cdot)$ has diameter $\leq 20L$.*

Note: This $\Rightarrow$ the Urysohn 1-width of $(\tilde{N}, g)$ is less than $20L$.

Proof of Theorem 2

Step 1 (covering argument): the above Proposition $\Rightarrow$ each finitely generated subgroup G of $\pi_1(N)$ cannot have *one end*.

Proof of Theorem 2

Step 1 (covering argument): the above Proposition $\Rightarrow$ each finitely generated subgroup G of $\pi_1(N)$ cannot have *one end*.

Inspired by a work of Ramachandran-Wolfson (2010).

Proof of Theorem 2

Step 1 (covering argument): the above Proposition $\Rightarrow$ each finitely generated subgroup G of $\pi_1(N)$ cannot have *one end*.

Inspired by a work of Ramachandran-Wolfson (2010).

Step 2 (geometric group theory): Classical works of Serre, Stalling, Dunwoody $\Rightarrow$ $\pi_1(N)$ is virtually free, i.e. it contains a finite index subgroup which is a free group.

Proof of Theorem 2

Step 1 (covering argument): the above Proposition $\Rightarrow$ each finitely generated subgroup G of $\pi_1(N)$ cannot have *one end*.

Inspired by a work of Ramachandran-Wolfson (2010).

Step 2 (geometric group theory): Classical works of Serre, Stalling, Dunwoody $\Rightarrow$ $\pi_1(N)$ is virtually free, i.e. it contains a finite index subgroup which is a free group.

Step 3: apply a purely topological result by Gadgil-Seshadri (2009): if N^n is a closed manifold, $\pi_1(N)$ is a free group with k generators, $\pi_2(N) = \cdots = \pi_{[\frac{n}{2}]}(N) = 0$, then N is homotopy equivalent to the connected sums of k copies of $S^{n-1} \times S^1$.

Thanks!

Thanks!