A new approach to classifying C^*-algebras

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For a C^*-algebra A let $\mathrm{Fin}(A, A)$ be the set of completely positive maps factoring through finite dimensional C^*-algebras. A is nuclear if id_A is in the point-norm closure of $\mathrm{Fin}(A, A)$. Equivalently, A is nuclear if and only if it has a unique C^*-tensor product with any other C^*-algebra.
A von Neumann algebra M is semidiscrete if id_M is in the point-ultraweak closure of $\mathrm{Fin}(M, M)$.
$M \subseteq \mathcal{B}(\mathcal{H})$ is injective if it admits a conditional expection.

Theorem (Connes)

The following are equivalent for a von Neumann algebra M:

- *M is injective;*
- *M is semidiscrete.*

If M has separable predual these are equivalent to

- *M is AFD: there exist finite dimensional C^*-subalgebras $F_1 \subseteq F_2 \subseteq \cdots \subseteq M$ such that $\overline{\bigcup F_n}^{uw} = M$.*

For a C^*-algebra A there is a universal representation $\pi_u\colon A \to \mathcal{B}(\mathcal{H}_u)$ (the direct sum of all GNS representations). Then $\pi_u(A)'' \cong A^{**}$ canonically.
On A the weak topology coincides with the ultraweak(=weak*) topology on A^{**}.

Lemma

If A^{**} is semidiscrete then A is nuclear.

Proof.

$\mathrm{Fin}(A, A)$ is convex and thus its point-norm closure and its point-weak closure agree by the Hahn–Banach theorem.
$\mathrm{id}_{A^{**}}$ is in the point-ultraweak closure of $\mathrm{Fin}(A^{**}, A^{**})$.
Hence id_A is in the point-weak closure of $\mathrm{Fin}(A, A)$. □

Corollary (Connes)

A is nuclear if and only if A^{**} is injective.

Let B be a C^*-algebra with a tracial state τ_B.
For a free ultrafilter ω on $\mathbb{N}$, we let

$$B_\omega := \frac{\{(b_k)_{k\in\mathbb{N}} \in \prod_{\mathbb{N}} B : \|b_k\| \text{ is bounded}\}}{\{(b_k)_k : \lim_{k\to\omega} \|b_k\| = 0\}}.$$

Then $\tau_\omega \colon B_\omega \to \mathbb{C}$ given by $\tau_\omega((b_k)_k) = \lim_{k\to\omega} \tau_B(b_k)$ is a tracial state.
The left kernel of τ_ω

$$J_{\tau_B} = \{(b_k)_k \in B_\omega : \lim_{k\to\omega} \tau_B(b_k^* b_k) = 0\}$$

is a two-sided closed ideal in B_ω.

Proposition (Folklore)

B_ω / J_{τ_B} is a von Neumann algebra and τ_ω induces a faithful normal tracial state on B_ω / J_{τ_B}.
In fact, $B_\omega / J_{\tau_B} \cong M^\omega$ where $M = \pi_{\tau_B}(B)''$.

Let B be a unital C^*-algebra with a tracial state τ_B. Assume B has no quotients of type I. Let A be a separable nuclear C^*-algebra with a faithful tracial state τ_A.

Corollary (Connes)

There exists a $*$-homomorphism $\phi\colon A \to B_\omega / J_{\tau_B}$ such that $\tau_A = \tau_\omega \circ \phi$.

Proof.

As B has no type I quotients, B_ω / J_{τ_B} is a type II_1 von Neumann algebra.
$\pi_{\tau_A}(A)''$ is injective with separable predual (since A is nuclear and separable). Hence $\pi_{\tau_A}(A)'' = \overline{\bigcup F_n}^{uw}$ by Connes.
By basic theory of II_1 von Neumann algebras there is a $*$-homomorphism $\phi\colon \bigcup F_n \to B_\omega / J_{\tau_B}$ such that $\tau_A''|_{\bigcup F_n} = \tau_\omega \circ \phi$.
Extend $\phi\colon \pi_{\tau_A}(A)'' \to B_\omega / J_{\tau_B}$ and restrict back to A. □

Question: when can we produce a $*$-homomorphism $\Phi\colon A \to B_\omega$ such that $\tau_A = \tau_\omega \circ \Phi$?
By Connes' theorem we can instead try to solve the lifting problem

$$\begin{array}{ccc} & & B_\omega \\ & \overset{\exists \Phi?}{\nearrow} & \downarrow \pi \\ A & \xrightarrow{\ \phi\ } & B_\omega / J_{\tau_B}. \end{array}$$

Theorem (Schafhauser)

If (B, τ_B) is very well-behaved (e.g. B is a unital, monotracial AF C^*-algebra, or an irrational rotation algebra) then it suffices to find a lift in KK-theory.
More precisely, for every $\kappa \in KK(A, B_\omega)$ such that $\pi_*(\kappa) = [\phi]_{KK}$, there exists a $*$-hom. $\Phi\colon A \to B_\omega$ such that $\pi \circ \Phi = \phi$ and $[\Phi]_{KK} = \kappa$.

A satisfies the UCT if for every C^*-algebra D there is a short exact sequence

$$\mathrm{Ext}(K_*(A), K_{1-*}(D)) \rightarrowtail KK(A, D) \twoheadrightarrow \mathrm{Hom}(K_*(A), K_*(D)).$$

Morally: the UCT means that KK-theory can be determined by K-theory.
Open problem: Do all separable nuclear C^*-algebras satisfy the UCT?

Theorem (Schafhauser)

If (B, τ_B) is very well-behaved (e.g. B is a unital, monotracial AF C^*-algebra, or an irrational rotation algebra) then for every $\kappa \in KK(A, B_\omega)$ such that $\pi_*(\kappa) = [\phi]_{KK}$, there exists a $*$-hom. $\Phi\colon A \to B_\omega$ such that $\pi \circ \Phi = \phi$ and $[\Phi]_{KK} = \kappa$.
If A satisfies the UCT, then for every homomorphism $\kappa_*\colon K_*(A) \to K_*(B_\omega)$ such that $\pi_* \circ \kappa_* = \phi_*$ there is a $*$-hom. $\Phi\colon A \to B_\omega$ such that $\pi \circ \Phi = \phi$ and $\Phi_* = \kappa_*$.

Take-away

The question of finding a trace-preserving $*$-hom. $\Phi\colon A \to B_\omega$ is broken into two parts:

- (Tracial von Neumann algebras): Find a trace-preserving $*$-homomorphism $\phi\colon A \to B_\omega / J_{\tau_B}$
- (KK-theory): Find a lift of ϕ in $KK(A, B_\omega)$.

First part is solved by Connes' theorem, the second part is solved using the UCT.

Theorem (Tikuisis–White–Winter)

Let A be a separable nuclear C^*-algebra satisfying the UCT. If A admits a faithful tracial state τ_A, then A is quasidiagonal.

Voiculescu: A is QD $\Leftrightarrow A \hookrightarrow B_\omega$ for an AF algebra B.
Schafhauser's proof of TWW: there is a homomorphism $(\tau_A)_*\colon K_0(A) \to \mathbb{R}$ induced by evaluating projections at τ_A.
There exists a simple unital monotracial AF algebra B (not type I) such that $(\tau_A)_*(K_0(A)) \subseteq K_0(B) \subseteq \mathbb{R}$.
Find $\phi\colon A \to B_\omega/J_{\tau_B} \cong \mathcal{R}^\omega$ (by Connes) such that $\tau_\omega \circ \phi = \tau_A$.
The composition $K_0(A) \xrightarrow{(\tau_A)_*} K_0(B) \to K_0(B_\omega)$ is such a lift of $\phi_0 = (\tau_A)_*\colon K_0(A) \to K_0(\mathcal{R}^\omega) = \mathbb{R}$.
By Schafhauser's theorem there is a $*$-homomorphism $\Phi\colon A \to B_\omega$ so that $\tau_\omega \circ \Phi = \tau_A$. This implies A is QD.

Theorem (Schafhauser)

With A and B as above, $A \hookrightarrow B \otimes \bigotimes_{n\in\mathbb{N}} M_n(\mathbb{C})$.

Theorem (Elliott–Gong–Lin–Niu, TWW, Kirchberg, Phillips,...)

Let A and B be separable nuclear simple unital $\mathcal{Z}$-stable C^*-algebras satisfying the UCT. Then $A \cong B$ if and only if A and B have the same K-theory and traces.

New approach (j.w. Carrión, Schafhauser, Tikuisis, White).
Very rough sketch when A and B have unique trace:
(Tracial part): find $\phi\colon A \to B_\omega / J_{\tau_B}$ trace-preserving.
(K-theory part): Let $\kappa_*\colon K_*(A) \to K_*(B)$ be an isomorphism (compatible with traces).
Arguing as in the previous slide, we lift ϕ to a $*$-hom.
$\Phi\colon A \to B_\omega$ such that $\tau_\omega \circ \Phi = \tau_A$ and
$\Phi_* = \iota_* \circ \kappa_*\colon K_*(A) \to K_*(B_\omega)$.
By working hard, this can be turned into a $*$-hom. $A \to B$ and then an isomorphism $A \xrightarrow{\cong} B$.

What if our C^*-algebras do not have a unique trace?
Let $T(B)$ be the set of tracial states and

$$J_{T(B)} := \{(b_k)_k \in B_\omega : \lim_{k\to\omega} \sup_{\tau\in T(B)} \tau(b_k^* b_k) = 0\}.$$

Tracial part: Then $B_\omega/J_{T(B)}$ behaves like a bundle of tracial von Neumann algebras. Using Connes' theorem and a gluing technique, we find a trace-preserving $*$-homomorphism $\phi\colon A \to B_\omega/J_{T(B)}$.
K-theory part: Schafhauser's lifting technique still applies in this case, so we can lift ϕ using the UCT and K-theory.

Theorem (Rosenberg, Tikuisis–White–Winter)

A discrete group G is amenable if and only if the left regular C^*-algebra $C^*_\lambda(G)$ is quasidiagonal.

Theorem (G)

Let G be a second countable, locally compact, unimodular group. If one of the following hold:

- *G is amenable,*
- *G is type I, or*
- *G is almost connected (i.e. G/G_0 is connected),*

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Idea: $L(G)$ is a semifinite von Neumann algebra with canonical semifinite trace (since G is unimodular).
One can find an AF algebra B and closed ideals $J \subseteq I \subseteq B_\omega$ such that the multiplier C^*-algebra $\mathcal{M}(I/J)$ is a II_∞-factor.
By Connes' theorem we can map $L(G) \to \mathcal{M}(I/J)$ such that it takes $C^*_\lambda(G) \to I/J$.
By modifying Schafhauser's theorem we lift to $C^*_\lambda(G) \to I \subseteq B_\omega$.

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then $C_\lambda^*(G)$ is quasidiagonal.

Question

If G is unimodular and $L(G)$ is injective, is $C_\lambda^*(G)$ quasidiagonal?

Converse is not true: $C_\lambda^*(G \times \mathbb{R})$ is QD for all G by Voiculescu, but $L(G \times \mathbb{R})$ is injective if and only if $L(G)$ is.