On Gromov's dihedral extremality and rigidity conjectures

Zhizhang Xie
(joint work with Jinmin Wang and Guoliang Yu)

Texas A&M University

Global NCG, Feb 2022

Gromov's dihedral extremality conjecture

Conjecture (Gromov's dihedral extremal conjecture for convex polyhedra)

Let P be a convex polyhedron in $\mathbb{R}^n$ and g_0 the Euclidean metric on P. If g is a Riemannian metric on P such that

1. *(scalar curvature comparison)* $\mathrm{Sc}(g) \geq \mathrm{Sc}(g_0) = 0$,
2. *(mean curvature comparison)* $H_g(F_i) \geq H_{g_0}(F_i) = 0$ *for each face* F_i *of* P*, and*
3. *(dihedral angle comparison)* $\theta_{ij}(g) \leq \theta_{ij}(g_0)$ *on each* $F_{ij} = F_i \cap F_j$,

then we have

$$\mathrm{Sc}(g) = 0, H_g(F_i) = 0 \text{ and } \theta_{ij}(g) = \theta_{ij}(g_0)$$

for all i and all $j \neq i$.

Gromov's dihedral rigidity conjecture

Conjecture (Gromov's dihedral rigidity conjecture for convex polyhedra)

Let P be a convex polyhedron in $\mathbb{R}^n$ and g_0 the Euclidean metric on P. If g is a smooth Riemannian metric on P such that

1. $\mathrm{Sc}(g) \geq \mathrm{Sc}(g_0) = 0$,
2. $H_g(F_i) \geq H_{g_0}(F_i) = 0$ *for each face* F_i *of* P, *and*
3. $\theta_{ij}(g) \leq \theta_{ij}(g_0)$ *on each* $F_{ij} = F_i \cap F_j$,

then g is also a flat metric.

Gauss-Bonnet Theorem

Theorem (Gauss-Bonnet)

Suppose (X, g) is an oriented two dimensional surface with piecewise smooth boundary. Then

$$2\pi\chi(X) = \int_X KdA + \sum \int_{C_i} H_g(s)ds + \sum \alpha_i$$

where α_i is the signed exterior angle at the intersection of C_i and C_{i+1}.

Dihedral angle $\theta_i = \pi - \alpha_i$.

$$2\pi\chi(X) = \int_X KdA + \sum \int_{C_i} H_g(s)ds - \sum \theta_i + \pi \cdot \#\text{vertices}$$

Theorem (Wang-X-Yu 2021)

Let P be a convex polyhedron in $\mathbb{R}^n$ and g_0 the Euclidean metric on P. If g is a smooth Riemannian metric on P such that

1. $\mathrm{Sc}(g) \geq \mathrm{Sc}(g_0) = 0$,
2. $H_g(F_i) \geq H_{g_0}(F_i) = 0$ *for each face* F_i *of* P, *and*
3. $\theta_{ij}(g) \leq \theta_{ij}(g_0)$ *on each* $F_{ij} = F_i \cap F_j$,

then we have

$$\mathrm{Sc}(g) = 0, H_g(F_i) = 0 \text{ and } \theta_{ij}(g) = \theta_{ij}(g_0)$$

for all i and all $j \neq i$. Furthermore, g is Ricci flat. Consequently, g is flat if $\dim P = 3$.

A little bit of history

1. Gromov showed that the dihedral extremal conjecture holds for a special class of convex polyhedra, under some extra restrictions on dihedral angles. For example, the standard Euclidean cube $[0,1]^n$.
2. More recently, Li proved the dihedral rigidity conjecture for a special class of convex polyhedra, which includes:
    1. cone type 3-dimensional polyhedra satisfying certain angle conditions,
    2. prism-type polyhedra of dimension ≤ 7, that is, P is the Cartesian product $P_0 \times [0,1]^{n-2}$, where $P_0 \subset \mathbb{R}^2$ is a 2-dimensional polygon with non-obtuse dihedral angles.

Implications of the conjectures

Gromov's dihedral extremal (or rigidity) conjecture can be viewed as a localization of the positive mass theorem.

Theorem (Positive Mass theorem, Schoen-Yau 1979, Witten 1981)

If (X, g) is a complete asymptotically Euclidean manifold of dimension $n \geq 3$ such that its scalar curvature is non-negative, then the ADM mass of each end of X is non-negative.

Schoen-Yau: $\dim X \leq 7$ but X not necessarily spin
Witten: X is spin but for all $n = \dim X \geq 3$

A more general comparison theorem

Theorem (Wang-X-Yu)

Let $(N, \overline{g})$ and (M, g) be compact oriented Riemannian manifolds with corners. Suppose

1. *the curvature operator of g is non-negative,*
2. *each codimension one face F_i of M is convex, that is, the second fundamental form of F_i is non-negative,*
3. *all dihedral angles $\theta_{ij}(g)$ of M are $\leq \pi$.*

Let $f : (N, \overline{g}) \to (M, g)$ be a spin corner map of manifolds with corners:

(1) *f is area-non-increasing on N, and f is distance-non-increasing on the boundary ∂N,*

(2) $\mathrm{Sc}(\overline{g})_x \geq \mathrm{Sc}(g)_{f(x)}$ *for all $x \in N$,*

(3) $H_{\overline{g}}(\overline{F}_i)_y \geq H_g(F_i)_{f(y)}$ *for all y in each codimension one face $\overline{F}_i$ of N,*

(4) $\theta_{ij}(\overline{g})_z \leq \theta_{ij}(g)_{f(z)}$ *for all $\overline{F}_i, \overline{F}_j$ and all $z \in \overline{F}_i \cap \overline{F}_j$,*

A general comparison theorem (continued)

(5) M has nonzero Euler characteristic,

(6) the $\widehat{A}$-degree $\deg_{\widehat{A}}(f)$ of f is nonzero, that is,

$$\deg_{\widehat{A}}(f) := \int_N \widehat{A}(N) \wedge f^*[M] \neq 0.$$

Then we have

1. $\mathrm{Sc}(\overline{g})_x = \mathrm{Sc}(g)_{f(x)}$ for all $x \in N$,
2. $H_{\overline{g}}(\overline{F}_i)_y = H_g(F_i)_{f(y)}$ for all $y \in \overline{F}_i$,
3. $\theta_{ij}(\overline{g})_z = \theta_{ij}(g)_{f(z)}$ for all $\overline{F}_i, \overline{F}_j$ and all $z \in \overline{F}_i \cap \overline{F}_j$,

and the following are true.

(i) If $\mathrm{Ric}(g) > 0$ and f is distance-non-increasing on the whole N, then f is a Riemannian submersion. Here $\mathrm{Ric}(g)$ is the Ricci curvature of the metric g on M.

(ii) If $0 < \mathrm{Ric}(g) < \frac{1}{2}\mathrm{Sc}(g) \cdot g$, then f is a Riemannian submersion.

(iii) If (M, g) is flat, then $(N, \overline{g})$ is Ricci flat. Consequently, $(N, \overline{g})$ is flat if $\dim N = 3$.

Some previous results

Theorem (Goette and Semmelmann 2002)

Comparison theorem for closed manifolds (i.e. no boundary).

Theorem (Lott 2021)

Comparison theorem for manifolds with smooth boundary (i.e. no dihedral angles).

$f: N \to M$ is a spin map, that is, the bundle $TN \oplus f^*TM$ admits a spin structure.

We shall consider the (twisted) Dirac operator on $S_N \otimes f^*S_M$.

$$D = \sum_i c(e_i)\nabla_{e_i}$$

Comparison of scalar curvature and mean curvature on manifolds with smooth boundary

Proposition (Goette-Semmelmann 2002, Lott 2021)

Let (M, g) and $(N, \overline{g})$ be two oriented compact Riemannian manifolds with smooth boundary and $f : N \to M$ is a spin map. Assume that both the curvature operator of M and the second fundamental form of ∂M are non-negative. Then for a smooth section φ of $S_N \otimes f^*S_M$ over N, we have

$$\int_N |D\varphi|^2 \geq \int_N |\nabla\varphi|^2 + \int_N \frac{\overline{\mathrm{Sc}}}{4}|\varphi|^2 - \int_N \| \wedge^2 df\| \cdot \frac{f^*\mathrm{Sc}}{4}|\varphi|^2$$
$$+ \int_{\partial N} \langle D^\partial \varphi, \varphi\rangle + \int_{\partial N} \frac{\overline{H}}{2}|\varphi|^2 - \int_{\partial N} \|df\| \cdot \frac{f^*H}{2}|\varphi|^2.$$

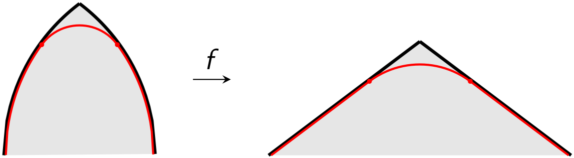
Comparison of scalar curvature, mean curvature and dihedral angle on manifolds with corners

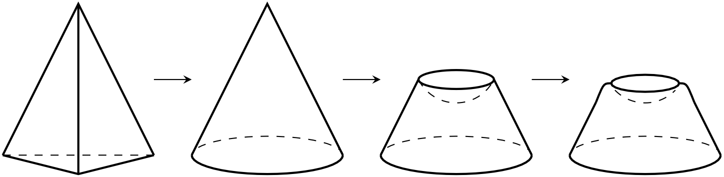
Proposition (Wang-X-Yu)

Let (M, g) and $(N, \overline{g})$ be two oriented compact Riemannian manifolds with corners. Suppose we are given a spin map $f: N \to M$ that is also a corner map. Let D be the Dirac operator on $S_N \otimes f^*S_M$. If both the curvature operator of M and the second fundamental form of ∂M are non-negative, then we have

$$\begin{aligned}\int_N |D\varphi|^2 \geq & \int_N |\nabla\varphi|^2 + \int_N \frac{\overline{\mathrm{Sc}}}{4}|\varphi|^2 - \int_N \| \wedge^2 df\| \cdot \frac{f^*\mathrm{Sc}}{4}|\varphi|^2 \\ & + \int_{\partial N} \langle D^\partial \varphi, \varphi\rangle + \int_{\partial N} \frac{\overline{H}}{2}|\varphi|^2 - \int_{\partial N} \|df\| \cdot \frac{f^*H}{2}|\varphi|^2 \\ & + \frac{1}{2}\sum_{i,j}\int_{\overline{F}_{ij}} (\pi - \overline{\theta}_{ij}) \cdot |\varphi|^2 - \frac{1}{2}\sum_{i,j}\int_{\overline{F}_{ij}} |\pi - \theta_{ij}| \cdot |\varphi|^2\end{aligned}$$

for all smooth sections φ of $S_N \otimes f^*S_M$.


f



Fredholm index of D with local boundary condition

Theorem (Wang-X-Yu)

Let D_B be the Dirac operator D on $S_N \otimes f^*S_M$ subject to the boundary condition B induced by $(\overline{\omega} \otimes \omega)(\overline{c}(\overline{e}_n) \otimes c(e_n))$, whose domain is

$$\mathrm{dom}(D_B) = C_0^\infty(N, S_N \otimes f^*S_M; B).$$

If the dihedral angles $\theta_{ij}(\overline{g})$ of N and $\theta_{ij}(g)$ of M satisfy

$$\theta_{ij}(\overline{g})_z \leq \theta_{ij}(g)_{f(z)} \leq \pi$$

for all codimension one faces $\overline{F}_i, \overline{F}_j$ of N and all $z \in \overline{F}_i \cap \overline{F}_j$, then D_B is an essentially self-adjoint Fredholm operator with index

$$\mathrm{Ind}(\overline{D}_B) = \deg_{\widehat{A}}(f) \cdot \chi(M),$$

where $\deg_{\widehat{A}}(f)$ is $\widehat{A}$-degree of f and $\chi(M)$ is Euler characteristic of M.

Remarks

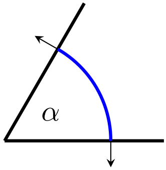
In order to compute the Fredholm index of D_B, we continuously deform manifolds with corners to manifolds with smooth boundary. The essential self-adjointness of D_B is a key step in proving the Fredholm index of D_B remains constant under this deformation. In a previous version of our paper (arXiv:2112.01510v2), we claimed such a homotopy invariance of Fredholm index without showing the essential self-adjointness of D_B. This has been corrected with a detailed proof in the new version of our paper (arXiv:2112.01510v3).

The dihedral angle condition

$$\theta_{ij}(\overline{g})_z \leq \theta_{ij}(g)_{f(z)} \leq \pi$$

is a sufficient and (almost) necessary condition for D_B to be essentially self-adjoint.

Essential self-adjointness of D_B



The de Rham operator (written in cylindrical coordinates) is

$$\begin{pmatrix} 0 & -\frac{\partial}{\partial r} \\ \frac{\partial}{\partial r} & 0 \end{pmatrix} + \frac{1}{r} \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}$$

where

$$P = \begin{pmatrix} -1/2 & -\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \theta} & -1/2 \end{pmatrix}$$

Essential self-adjointness of D_B

The boundary condition B for P is that

$$\phi_1(0) = \phi_1(\alpha) = 0 \text{ for } \phi = \phi_0(\theta) + \phi_1(\theta)d\theta.$$

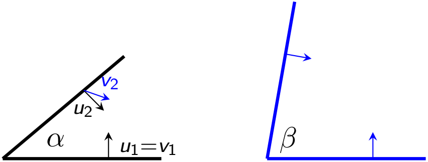
The spectrum of P subject to the boundary condition B is

$$\Big\{ -\frac{1}{2} + \frac{k\pi}{\alpha} \Big\}_{k\in\mathbb{Z}}.$$

Lemma (Cheeger, Chou, Brüning-Seeley, $\cdots$)

Let P_B be the operator P on the link subject to the induced boundary B. Assume that P_B is essentially self-adjoint. Then D_B^{dR} is essentially self-adjoint $\Leftrightarrow$ the deficiency indices of D_B^{dR} are zero $\Leftrightarrow$ $|P_B| \geq 1/2$. Here the deficiency indices of D_B^{dR} are codim Ran$(D_B^{dR} \pm i)$.

Essential self-adjointness of D_B



The boundary condition B for P in this case is following: $\phi_1(0) = 0$, and

$$-\phi_0(\alpha)\sin(\frac{\beta-\alpha}{2}) + \phi_1(\alpha)\cos(\frac{\beta-\alpha}{2}) = 0$$

for $\phi = \phi_0(\theta) + \phi_1(\theta)d\theta$.

The spectrum of P_B is $\left\{ -\frac{\beta}{2\alpha} + \frac{k\pi}{\alpha} \right\}_{k\in\mathbb{Z}}$.

$|P_B| \geq 1/2$ if and only if $(\alpha+\beta \leq 2\pi$ and $\alpha \leq \beta)$.

Essential self-adjointness of D_B

Higher dimensional case

The higher dimensional case is proved by induction together with the following Gallot–Meyer inequality for the Hodge Laplacian

$$\Delta = \nabla^*\nabla + \mathcal{R}_p \geq \nabla^*\nabla + p(n-1-p)\lambda_x$$

on p-forms, where λ_x is the (pointwise) minimal eigenvalue of the curvature operator. On the round sphere of dimension ≥ 2, we have $\lambda_x \equiv 1$.

Thank you!