Morita equivalence for operator systems

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Outline

(1) Origins and samples of Morita equivalence

(2) Morita and Rieffel

(3) Operator systems – first *why*'s and *how*'s

(4) Δ-equivalence and stable isomorphism

(5) A viewpoint from non-commutative graph theory

(6) Transference of approximation properties

(7) Δ-equivalence of graph operator systems

Concrete vs. *abstract*

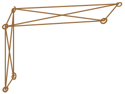
Abstract object	Representation	
Group G	Homomorphism $\pi : G \to \mathrm{GL}(V)$	V vector space
Ring R	Left R-module M	
Algebra A	Homomorphism $\pi : A \to \mathrm{End}(V)$	V vector space
C*-algebra $\mathcal{A}$	*-representation $\pi : \mathcal{A} \to \mathcal{B}(H)$	H Hilbert space

Main idea: Look at the concrete manifestations of the abstract object instead of the object itself.

Note: Often we only have access to those concrete manifestations (cf. *Plato*)

$\rightsquigarrow$ Morita equivalence: A and B have equivalent representation theories.

Rings: R and $M_n(R) = R \otimes M_n$

Graphs:  

The Morita theorems (1950's)

Notation: $R \sim S$ designates that R and S are Morita equivalent. R and S rings

Equivalent views on $R \sim S$:

- $\exists$ functors $\mathcal{F} : \mathrm{Mod}_R \to \mathrm{Mod}_S$ and $\mathcal{G} : \mathrm{Mod}_S \to \mathrm{Mod}_R$ s.t. $\mathcal{G} \circ \mathcal{F} \cong \mathrm{id}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathrm{id}$;
- $\exists$ ${}_S X_R$ and ${}_R Y_S$ s.t.

$$R = Y \otimes_S X \quad \text{and} \quad S = X \otimes_R Y;$$

- $\exists$ ${}_S X_R$ and ${}_R Y_S$ and balanced module maps

$$(\cdot,\cdot) : Y \times X \to R, \quad [\cdot,\cdot] : X \times Y \to S$$

  satisfying associativity;
- $\mathrm{End}(R^{\mathbb{N}}) \cong \mathrm{End}(S^{\mathbb{N}})$ (stable isomorphism, Camillo, 1984).

C*-algebras: the work of Rieffel and its aftermath

Strong Morita equivalence of C*-alg. $\mathcal{A}$ and $\mathcal{B}$:

$$\leftrightsquigarrow \quad \exists \text{ linking algebra } \mathcal{L} = \begin{pmatrix} \rho(\mathcal{B}) & \mathcal{M} \\ \mathcal{M}^* & \pi(\mathcal{A}) \end{pmatrix}$$

Brief summary: Identical to all but the equivalence of the rep. theories.

> Question: Other categories? – non-selfadjoint operator algebras, operator spaces, operator systems?

Blecher, Muhly, Paulsen, Kashyap et al

Motivation: Ring-theoretic setting
$\leftrightsquigarrow$ no adjoint operation available.

$$\mathcal{Y}\mathcal{B}\mathcal{X} \subseteq \mathcal{A} \text{ and } \mathcal{X}\mathcal{A}\mathcal{Y} \subseteq \mathcal{B}$$

$\mathcal{X}$ and $\mathcal{Y}$ operator spaces

Eleftherakis, *Kakariadis, Paulsen, T. et al*

Motivation: Use selfadjoint-ness
(as much as possible).

$$\mathcal{M}^*\mathcal{B}\mathcal{M} \subseteq \mathcal{A} \text{ and } \mathcal{M}\mathcal{A}\mathcal{M}^* \subseteq \mathcal{B}$$

$\mathcal{M}$ a ternary ring of operators (TRO).

A TRO is a subspace $\mathcal{M} \subseteq \mathcal{B}(H, K)$ satisfying $\mathcal{M}\mathcal{M}^*\mathcal{M} \subseteq \mathcal{M}$.

(Hilbert bimodules, imprimitivity bimodules, corners in C*-alg.: *Paschke, Zettl, Harris, Katavolos-T, Blecher, Ruan et al*)

Operator systems – examples

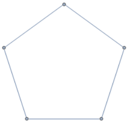
A unital C*-algebra: closed subalgebra $\mathcal{A} \subseteq \mathcal{B}(H)$ s.t. $I \in \mathcal{S}$ and $T \in \mathcal{S} \Rightarrow T^* \in \mathcal{S}$.

An operator system: subspace $\mathcal{S} \subseteq \mathcal{B}(H)$ s.t. $I \in \mathcal{S}$ and $T \in \mathcal{S} \Rightarrow T^* \in \mathcal{S}$.

Philosophy: Op. systems describe phenomena under conditions of partial knowledge.

- Graph op. systems: G a graph on n vertices $\rightsquigarrow$ $\mathcal{S}_G = \mathrm{span}\{\epsilon_{i,j} : i \simeq j\}$.

  $\epsilon_{i,j}$: matrix units in M_n, $i \simeq j$: adjacency/equality



$$\mathcal{S}_{C_5} = \begin{pmatrix} * & * & 0 & 0 & * \\ * & * & * & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & * & * & * \\ * & 0 & 0 & * & * \end{pmatrix}$$

$\rightsquigarrow$ positive completion problem (*Paulsen-Power-Smith*)

$\leftrightsquigarrow$ tolerance relations (*Connes-van Suijlekom*)

- Feijér-Riesz op. system (*Connes-van Suijlekom*): $\mathcal{F}_n = \{f \in C(\mathbb{T}) : \mathrm{supp} \hat{f} \subseteq [-n, n]\}$
- The Paulsen system: $\mathcal{X} \subseteq \mathcal{B}(H, K)$ $\rightsquigarrow$ $\mathcal{S}_{\mathcal{X}} := \begin{pmatrix} \mathbb{C}I & \mathcal{X} \\ \mathcal{X}^* & \mathbb{C}I \end{pmatrix}$

  Allows to reduce the study of $\mathcal{X}$ to the study of $\mathcal{S}_{\mathcal{X}}$ (Note: $\|x\| \leq 1$ iff $\begin{pmatrix} I & x \\ x^* & I \end{pmatrix} \geq 0$)

Operator systems – concrete and abstract

$\mathcal{S} \subseteq \mathcal{B}(H)$ op. sys. $\rightsquigarrow$ $M_n(\mathcal{S}) \subseteq \mathcal{B}(H^n)$ $\rightsquigarrow$ $M_n(\mathcal{S})^+ \subseteq \mathcal{B}(H^n)^+$

and $A^* M_n(\mathcal{S})^+ A \subseteq M_m(\mathcal{S})^+$ $(A \in M_{n,m})$

$\rightsquigarrow$ I is an Archimedean matrix order unit $(x + rI \in \mathcal{S}^+ \forall r > 0 \Rightarrow x \in \mathcal{S}^+)$

$\varphi : \mathcal{S} \to \mathcal{T}$ $\rightsquigarrow$ $\varphi^{(n)} : M_n(\mathcal{S}) \to M_n(\mathcal{T})$, $\varphi^{(n)}((x_{i,j})) = (\varphi(x_{i,j}))$

φ completely positive if $\varphi^{(n)}(M_n(\mathcal{S})^+) \subseteq M_n(\mathcal{T})^+$, for every n.

$\rightsquigarrow$ complete order isomorphism (c.o.i.), when φ and φ^{-1} are c.p.

Choi-Effros' Theorem

If $(V, (C_n)_{n\in\mathbb{N}}, 1)$ an Archimedean matrix order unit space with

$$A^* \cdot C_n \cdot A \subseteq C_m, \quad A \in M_{n,m}, \ n, m \in \mathbb{N},$$

then $\exists$ c.o.i. $\varphi : V \to \mathcal{B}(H)$.

Usefulness: Tensor product theory, quotient theory

Note: $\exists$ canonical C*-algebras, associated with an operator system $\mathcal{S}$:

- C*-envelope $C^*(\mathcal{S})$: "smallest" C*-algebra generated by $\mathcal{S}$;
- C*-multiplier $\mathcal{A}_{\mathcal{S}}$: "largest" C*-algebra $\mathcal{A}$ with $\mathcal{A}\mathcal{S} \subseteq \mathcal{S}$.

TRO and Δ-equivalence

Definition

Operator systems $\mathcal{S} \subseteq \mathcal{B}(H)$ and $\mathcal{T} \subseteq \mathcal{B}(K)$ are called

- TRO-equivalent if $\exists$ non-degenerate TRO $\mathcal{M} \subseteq \mathcal{B}(H, K)$ s.t.

$$\mathcal{M}^*\mathcal{T}\mathcal{M} \subseteq \mathcal{S} \quad \text{and} \quad \mathcal{M}\mathcal{S}\mathcal{M}^* \subseteq \mathcal{T}.$$

Notation: $\mathcal{S} \sim_{\mathrm{TRO}} \mathcal{T}$

- Δ-equivalent if $\exists$ TRO-equivalent c.o.i. images of $\mathcal{S}$ and $\mathcal{T}$.

Notation: $\mathcal{S} \sim_{\Delta} \mathcal{T}$

Note:

- TRO-equivalence and Δ-equivalence are equivalence relations.
- The notion coincides with strong Morita equivalence if $\mathcal{S}$ and $\mathcal{T}$ are C*-algebras.

The following hold for operator systems $\mathcal{S} \subseteq \mathcal{B}(H)$ and $\mathcal{T} \subseteq \mathcal{B}(K)$:

- $\mathcal{S} \sim_{\Delta} \mathcal{T}$ iff $\mathcal{S} \otimes \mathfrak{K} \cong_{\mathrm{c.o.i.}} \mathcal{T} \otimes \mathfrak{K}$ (stable isomorphism)

$\mathfrak{K}$: compact operators on a separable Hilbert space

- $\mathcal{S} \sim_{\Delta} \mathcal{T}$ implies $C_e^*(\mathcal{S}) \sim_{\Delta} C_e^*(\mathcal{T})$ and $\mathcal{A}_{\mathcal{S}} \sim_{\Delta} \mathcal{A}_{\mathcal{T}}$.

Non-commutative graphs

A channel $\mathcal{N} : X \to Y$ is a family $\{(p(y|x))_{y\in Y} : x \in X\}$ of probability distributions over Y, one for each input symbol x.

The confusability graph $G_{\mathcal{N}}$ of $\mathcal{N}$: vertex set X, and adjacency

- $x \sim x'$ if $\mathrm{support}(p(\cdot|x)) \cap \mathrm{support}(p(\cdot|x')) \neq \emptyset$ ($\exists\ y \in Y$ s.t. $p(y|x) > 0$ and $p(y|x') > 0$)

Motivation (*Shannon*): A zero-error code for $\mathcal{N}$: a subset $A \subseteq X$ such that each symbol from A can be identified unambiguously after its receipt, despite the noise.

$\rightsquigarrow$ $A \subseteq X$ such that $\mathrm{support}(p(\cdot|x)) \cap \mathrm{support}(p(\cdot|x')) = \emptyset$ whenever $x, x' \in A$ distinct.

For a quantum channel $\Phi : M_d \to M_k$, $\Phi(T) = \sum_{p=1}^{r} A_p T A_p^*$, let

$$\mathcal{S}_\Phi = \mathrm{span}\{A_p A_q^* : p, q = 1, \dots, r\}.$$

non-commutative confusability graph of Φ

- $\mathcal{S}_\Phi$ is an operator subsystem of M_d;
- $\mathcal{S}_\Phi$ is independent of the Kraus representation of Φ.

Duan-Severini-Winter, 2013

- A non-commutative graph in M_d is an operator system in M_d;
- If Φ classical and $G = G_\Phi$ then $\mathcal{S}_\Phi = \mathcal{S}_{G_\Phi}$.

Arveson, Boreland, Brannan, Connes, Ganesan, Harris, Paulsen, van Suijlekom, T., Turowska, Voigt, Weaver, Winter et al

Note (*Paulsen*): $\mathcal{S}_{G_1} \cong_{\mathrm{c.o.i.}} \mathcal{S}_{G_2}$ iff $G_1 \cong G_2$

Non-commutative graph homomorphisms

Recall: A graph homom. $G \to H$ is $\varphi : V(G) \to V(H)$ s.t. $x \sim x' \Rightarrow \varphi(x) \sim \varphi(x')$.

Stahlke, 2016: If $\mathcal{S}$ and $\mathcal{T}$ NC graphs, a NC graph co-homom. $\mathcal{S} \to \mathcal{T}$ is a quantum channel,

$$\Phi(T) = \sum_{p=1}^{r} A_p T A_p^* \quad \text{with} \quad A_i^* \mathcal{T} A_j \subseteq \mathcal{S}, \ \ \forall\, i, j.$$

Verify: $\mathcal{S}_G \to \mathcal{S}_H$ iff $G^c \to H^c$.

With $\mathcal{X} = \mathrm{span}\{A_1, \ldots, A_r\}$, the relation $\mathcal{S} \to \mathcal{T}$ becomes

$$\mathcal{X}^* \mathcal{T} \mathcal{X} \subseteq \mathcal{S}$$

.

Observe: $\mathcal{X}^* \mathcal{T} \mathcal{X} \subseteq \mathcal{S}$ and $\mathcal{X} \mathcal{S} \mathcal{X}^* \subseteq \mathcal{T}$ yield $\mathcal{S} \sim_{\mathrm{TRO}} \mathcal{T}$.

Reason: $(\mathcal{X}\mathcal{X}^*\mathcal{X})\mathcal{T}(\mathcal{X}^*\mathcal{X}\mathcal{X}^*) \subseteq \mathcal{X}\mathcal{X}^*\mathcal{S}\mathcal{X}\mathcal{X}^* \subseteq \mathcal{X}\mathcal{T}\mathcal{X}^* \subseteq \mathcal{S}$

Bihomomorphism contexts

$\mathcal{S}$ and $\mathcal{T}$ operator systems, $\mathcal{X}$ a non-degenerate operator space.

- Abstract products: completely contractive completely positive

$$[\cdot,\cdot,\cdot]\colon \mathcal{X}^* \times \mathcal{T} \times \mathcal{X} \longrightarrow \mathcal{S}, \quad (\cdot,\cdot,\cdot)\colon \mathcal{X} \times \mathcal{S} \times \mathcal{X}^* \longrightarrow \mathcal{T}.$$

- Respecting diagonals: $[\mathcal{X}^*, 1_{\mathcal{T}}, \mathcal{X}] \subseteq \mathcal{A}_{\mathcal{S}}$ and $(\mathcal{X}, 1_{\mathcal{S}}, \mathcal{X}^*) \subseteq \mathcal{A}_{\mathcal{T}}$;
- Associativity:

$$[x_1^*, (x_2, s, x_3^*), x_4] = [x_1^*, 1_{\mathcal{T}}, x_2] \cdot s \cdot [x_3^*, 1_{\mathcal{T}}, x_4],$$

$$(x_1, [x_2^*, t, x_3], x_4^*) = (x_1, 1_{\mathcal{S}}, x_2^*) \cdot t \cdot (x_3, 1_{\mathcal{S}}, x_4^*)$$

- Non-degeneracy: $\lim_i [x_i^*, 1_{\mathcal{T}} \otimes I, y_i] = 1_{\mathcal{S}}$, $\lim_i (z_i, 1_{\mathcal{S}} \otimes I, w_i^*) = 1_{\mathcal{T}}$.

$\rightsquigarrow$ Bihomomorphism context

Note: If $\mathcal{S} \sim_{\Delta} \mathcal{T}$ then $\exists$ bihomomorphism context for $\mathcal{S}$ and $\mathcal{T}$.

Indeed: Assume WLOG $\mathcal{S} \sim_{\mathrm{TRO}} \mathcal{T}$, i.e.

$$\mathcal{M}^* \mathcal{T} \mathcal{M} \subseteq \mathcal{S} \quad \text{and} \quad \mathcal{M} \mathcal{S} \mathcal{M}^* \subseteq \mathcal{T}.$$

Set $\mathcal{X} = \mathcal{M}$, and

$$[x^*, t, y] = x^* t y \quad \text{and} \quad (x, s, y^*) = x s y^*.$$

Bihomomorphism contexts are good Morita contexts

Theorem

If $(\mathcal{S}, \mathcal{T}, \mathcal{X}, [\cdot,\cdot,\cdot], (\cdot,\cdot,\cdot))$ is a bihomomorphism context then $\mathcal{S} \sim_\Delta \mathcal{T}$.

Aim: Find concrete versions $\tilde{\mathcal{S}}$, $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{X}}$ with $\tilde{\mathcal{X}}^*\tilde{\mathcal{T}}\tilde{\mathcal{X}} \subseteq \tilde{\mathcal{S}}$ and $\tilde{\mathcal{X}}\tilde{\mathcal{S}}\tilde{\mathcal{X}}^* \subseteq \tilde{\mathcal{T}}$.

1) Make $\mathcal{S}$ concrete: Consider a c.o.i. embedding $\phi : \mathcal{S} \to \mathcal{B}(H)$.

2) Find the Hilbert space K for $\tilde{\mathcal{T}}$: Equip $\mathcal{X} \odot H$ with the sesqui-linear form

$$\langle x_1 \otimes h_1, x_2 \otimes h_2 \rangle := \langle \phi([x_2^*, 1_{\mathcal{T}}, x_1])h_1, h_2 \rangle_H .$$

Quotient out the kernel and complete to obtain K.

3) Represent $\mathcal{X}$ in $\mathcal{B}(H, K)$: $\theta : \mathcal{X} \to \mathcal{B}(H, K)$, $\theta(x)(h) = x \otimes h$.

4) Represent $\mathcal{T}$ on K: $\psi : \mathcal{T} \to \mathcal{B}(K)$,

$$\langle \psi(t)(x_1 \otimes h_1), x_2 \otimes h_2 \rangle_K = \langle \phi([x_2^*, t, x_1])h_1, h_2 \rangle_H .$$

To verify that ψ is c.o.i. use the complete positivity and non-degenerateness of $[\cdot,\cdot,\cdot]$.

5) Verify the concrete relations: $\theta(x_2)^*\psi(t)\theta(x_1) = \phi([x_2^*, t, x_1])$.

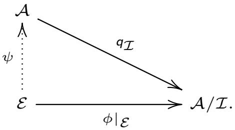
Preservation of approximation properties

Approximation by finite dim. structures or by canonical inner structures

$\rightsquigarrow$ approximation properties for C*-algebras and operator spaces.

For example: An operator system $\mathcal{S}$

- is exact if $\mathcal{A} \otimes \mathcal{S} / \mathcal{I} \otimes \mathcal{S} \to (\mathcal{A}/\mathcal{I}) \otimes \mathcal{S}$ is a c.o.i.; ($\otimes$ minimal)
- has local lifting property (OSLLP) if, for fin. dim. $\mathcal{E} \subseteq \mathcal{S}$,



Kavruk-Paulsen-T-Tomforde, 2011, 2013:

- Introduced op. syst. tensor products, $\mathcal{S} \otimes_{\max} \mathcal{T}$, $\mathcal{S} \otimes_{\rm c} \mathcal{T}$, $\mathcal{S} \otimes_{\rm er} \mathcal{T}$, $\mathcal{S} \otimes_{\rm el} \mathcal{T}$.
  - $\mathcal{S} \otimes_{\max} \mathcal{T}$ synthesised from within by $s \otimes t$, where $s \geq 0$ and $t \geq 0$;
  - $\mathcal{S} \otimes_{\rm c} \mathcal{T}$ determined from outside by $(\phi \cdot \psi)(u) \geq 0$ ($\phi : \mathcal{S} \to \mathcal{B}(H)$, $\psi : \mathcal{T} \to \mathcal{B}(H)$ comm. ranges);
  - $\mathcal{S} \otimes_{\rm el} \mathcal{T} \subseteq_{\rm c.o.i.} \mathcal{B}(H) \otimes_{\max} \mathcal{T}$, $\mathcal{S} \otimes_{\rm er} \mathcal{T} \subseteq_{\rm c.o.i.} \mathcal{S} \otimes_{\max} \mathcal{B}(K)$.
- Viewed approximation properties as nuclearity properties.
  - $\mathcal{S}$ is exact if and only if $\mathcal{S}$ is $(\min, {\rm el})$-nuclear, i.e. $\mathcal{S} \otimes \mathcal{T} = \mathcal{S} \otimes_{\rm el} \mathcal{T}$ for all $\mathcal{T}$;
  - $\mathcal{S}$ has the (OSLLP) if and only if $\mathcal{S}$ is $(\min, {\rm er})$-nuclear.

Assume $\mathcal{S} \sim_\Delta \mathcal{T}$ and τ, τ' "natural" tensor products

$\mathcal{S}$ (τ, τ')-nuclear $\Leftrightarrow$ $\mathcal{T}$ (τ, τ')-nuclear $\rightsquigarrow$ $\mathcal{S}$ exact (OSLLP) iff $\mathcal{T}$ exact (OSLLP) etc..

Key: Approximate c.p. factorisation of the identity map: $\mathcal{T} \to M_n(\mathcal{S}) \to \mathcal{T}$.

Equivalence of representation theories

Recall: An op.syst. $\mathcal{S}$ has a multiplier C*-algebra $\mathcal{A}_\mathcal{S} \subseteq \mathcal{S}$, with $\mathcal{A}_\mathcal{S}\mathcal{S} \subseteq \mathcal{S}$.

A C*-rep. of $\mathcal{S}$: (H, ϕ, π), $\phi\colon \mathcal{S} \to \mathcal{B}(H)$ u.c.p. map, $\pi : \mathcal{A}_\mathcal{S} \to \mathcal{B}(H)$ *-homom. with

$$\phi(a \cdot s) = \pi(a)\phi(s).$$

Category $\mathrm{Rep}_{C^*}(\mathcal{S})$

- Objects: C*-representations $\Gamma = (H, \phi, \pi)$;
- Morphisms: $\Gamma_1 \to \Gamma_2 \rightsquigarrow$ completely bounded maps $\Phi\colon \phi_1(\mathcal{S}) \to \phi_2(\mathcal{S})$ s.t.

$$\Phi(\phi_1(a \cdot s \cdot b)) = \Phi(\pi_1(a)\phi_1(s)\pi_1(b)) = \pi_2(a)\Phi(\phi_1(s))\pi_2(b).$$

Theorem

If $\mathcal{S} \sim_\Delta \mathcal{T}$ then the categories $\mathrm{Rep}_{C^*}(\mathcal{S})$ and $\mathrm{Rep}_{C^*}(\mathcal{T})$ are equivalent.

Aim: Define a functor $\mathcal{F}\colon \mathrm{Rep}_{C^*}(\mathcal{S}) \to \mathrm{Rep}_{C^*}(\mathcal{T})$.

1) Concreteness: Suppose that $\mathcal{S} \sim_{\mathrm{TRO}} \mathcal{T}$ via a TRO $\mathcal{M}$.

2) Images of objects: $(H, \phi, \pi) \in \mathrm{Rep}_{C^*}(\mathcal{S})$

$\rightsquigarrow$ K, the completion of $\mathcal{M} \otimes_{\mathcal{A}_\mathcal{S}} H$ w.r.t. $\langle a \otimes \xi, b \otimes \eta\rangle = \langle \phi(b^* a)\xi, \eta\rangle$.

$\rightsquigarrow$ ψ given by $\langle \psi(t)(a \otimes \xi), b \otimes \eta\rangle = \langle \phi(b^* \cdot t \cdot a)\xi, \eta\rangle$ $\rightsquigarrow$ σ given by $\sigma := \psi|_{\mathcal{A}_\mathcal{T}}$.

$\rightsquigarrow$ $\mathcal{F}((H_\phi, \phi, \pi)) = (H_\psi, \psi, \sigma)$.

Equivalence of representation theories – consequences

3) Rep. of $\mathcal{M}$: $\nu\colon \mathcal{M} \to \mathcal{B}(H,K)$, $\nu(m)(\xi) = m \otimes \xi \rightsquigarrow \nu(m)^*(n \otimes \xi) = \phi(m^* n)(\xi)$

4) Images of morphisms: Given $\Phi : (H_1, \phi_1, \pi_1) \to (H_2, \phi_2, \pi_2)$, define

$$\Psi\colon \psi_1(\mathcal{T}) \to \psi_2(\mathcal{T});\ \nu_1(n)\phi_1(s)\nu_1(m)^* \mapsto \nu_2(n)\Phi(\phi_1(s))\nu_2(m)^*.$$

Further consequences of $\mathcal{S} \sim_\Delta \mathcal{T} \leftrightsquigarrow$ quantised analysis in the sense of *Arveson*.

- Arveson (1969): operator systems are non-commutative function systems, i.e. $1 \in \mathcal{F} = \overline{\mathcal{F}} \subseteq C(X)$.
- Non-commutative version of the Shilov boundary? $\rightsquigarrow$
- Hamana (1979): $C_e^*(\mathcal{S})$ s.t. $\forall$ u.c.i. $\phi : \mathcal{S} \to \mathcal{A}$ $\exists$ *-epi $q : C^*(\phi(\mathcal{S})) \to C_e^*(\mathcal{S})$.
- Dritschel-McCullough (2005): $C_e^*(\mathcal{S}) = C^*(\phi(\mathcal{S}))$ if ϕ is maximal, i.e. non-dilatable non-trivially.

Theorem

Assume $\mathcal{S} \sim_\Delta \mathcal{T} \rightsquigarrow \mathcal{F}\colon \mathrm{Rep}_{C^*}(\mathcal{S}) \to \mathrm{Rep}_{C^*}(\mathcal{T})$ and $\mathcal{G}\colon \mathrm{Rep}_{C^*}(\mathcal{T}) \to \mathrm{Rep}_{C^*}(\mathcal{S})$.

$\Rightarrow$ (H,ϕ) maximal iff $\mathcal{F}((H,\phi))$ maximal.

$\rightsquigarrow$ ways to produce $C_e^*(\mathcal{S})$ yield ways to produce $C_e^*(\mathcal{T})$.

Interpretation: The less rigid that isomorphism Morita equivalence is still good for extremal analysis.

(Non-commutative) graphs revisited

Recall:

- NS graph is an operator system $\mathcal{S} \subseteq M_d$;
- For a graph G, we form the graph operator system $\mathcal{S}_G = \mathrm{span}\{\epsilon_{i,j} : i \simeq j\}$.

Question: What does $\mathcal{S} \sim_\Delta \mathcal{T}$ mean for non-commutative graphs?

Theorem

Let $\mathcal{S}$ and $\mathcal{T}$ be NC graphs. Then $\mathcal{S} \sim_\Delta \mathcal{T}$ iff $\mathcal{S} \sim_{\mathrm{TRO}} \mathcal{T}$.

Key: Ortiz-Paulsen: $C_e^*(\mathcal{S}) = C^*(\mathcal{S}) = \oplus_{i=1}^k M_{d_i}$ $+$ Δ-equivalence lifts to a TRO-equivalence of the C*-envelopes.

Theorem

$\mathcal{S}_G \sim_\Delta \mathcal{S}_H$ iff $\exists$ graph G' and H' s.t.

- $G' \cong H'$
- G pullback of G', and H pullback of H'.

- Pullback: $\exists\, f : V(G) \to V(G')$ s.t. $i \simeq_G j$ iff $f(i) \simeq_{G'} f(j)$.
- Key: $\mathcal{S}_G$ is a bimodule over the diagonal algebra (masa) $\mathcal{D}_d$, and
- the TRO's $\mathcal{M} \subseteq M_{d,k}$ that are masa-bimodules have the form

$$\mathcal{M} = \{T : T \text{ supported on } \{(i,j) : f(i) = g(j)\}\}.$$

Some further points

- For function systems, Δ-equivalence $\Leftrightarrow$ complete order isomorphism.

  Note: Extends a C*-algebra result: if $\mathcal{A}$ and $\mathcal{B}$ abelian then $\mathcal{A} \sim_\Delta \mathcal{B}$ iff $\mathcal{A} \cong \mathcal{B}$.

- Morita embedding $\mathcal{S} \subseteq_\Delta \mathcal{T}$: $\mathcal{S} \sim_\Delta p\iota(\mathcal{T})$, where $\iota : \mathcal{T} \to C_e^*(\mathcal{T})$ canonical inclusion and $p \in \mathcal{Z}(C_e^*(\mathcal{T})^{**})$ projection.

- $\mathcal{S} \subseteq_\Delta \mathcal{T}$ iff *-epi $C_e^*(\mathcal{T}) \otimes \mathfrak{K} \to C_e^*(\mathcal{S}) \otimes \mathfrak{K}$ mapping
$$\iota(\mathcal{T}) \otimes \mathfrak{K} \mapsto \iota(\mathcal{S}) \otimes \mathfrak{K}.$$

- Question: Relation with NC graph homomorphims? – Not examined.

- Question: Factorisation characterisation? $\rightsquigarrow$ Yes (available soon).

THANK YOU VERY MUCH!