Noncommutative geometry in Hochschild complexes

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Plan

Hochschild, (co)homology for an associative unital A

Hochschild chains $CH_n(A, M) = M \otimes A^{\otimes n} \rightsquigarrow HH_*(A, M)$

$$
\begin{aligned}
d_i(m \otimes a_1 \otimes \cdots \otimes a_n) &= m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\
d_0(m \otimes a_1 \otimes \cdots \otimes a_n) &= (m a_1 \otimes \cdots \otimes a_n) \\
d_n(m \otimes a_1 \otimes \cdots \otimes a_n) &= (a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1})
\end{aligned} \tag{1}
$$

$CH_*(A) = CH_*(A, A) = C^*(A)$

Hochschild cochains $CH^*(A, M) = Hom(A^{\otimes n}, M) \rightsquigarrow HH^*(A, M)$

$$
\begin{aligned}
d_i(f(a_1 \otimes \cdots \otimes a_n) &= f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\
d_0(f(a_1 \otimes \cdots \otimes a_n) &= a_1 f(a_2 \otimes \cdots \otimes a_n) \\
d_n(m \otimes a_1 \otimes \cdots \otimes a_n) &= f(a_1 \otimes \cdots \otimes a_{n-1}) a_n
\end{aligned} \tag{2}
$$

$CH^*(A) = CH^*(A, A) = C^*(A)$

Gerstenhaber [Ger63]

Operations on $CH^*(A,A)$

- Cup product:
  $f \cup g(a_1 \otimes \cdots \otimes a_{n+m}) = f(a_1 \otimes \cdots \otimes a_n) g(a_{n+1} \otimes \cdots \otimes a_{n+m})$
  Non–commutative if B is not commutative.
- Insertion $f \circ_i g(a_1 \otimes \cdots \otimes a_{n+m-1}) =$
  $f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m}) \otimes a_{i+m+1} \otimes \cdots \otimes a_{n+m-1})$
- Pre–Lie: $f \circ g = \sum (-1)^{i-1} f \circ_i g$
- Gerstenhaber bracket $\{f \bullet g\} = f \circ g \mp g \circ f$.
  Note this is Lie for degrees shifted by 1. It is also odd Poisson for $\cup$.

Theorem (Gerstenhaber)

$\cup$ is commutative on $HH^*(A)$. The homotopy is given by $\circ$. The operation $\{- \bullet -\}$ is a cohomology operation.

Deligne's conjecture

Operads

- The $\circ_i$ operations define a pseudo–operad structure. Their properties actually yield a definition. $\{\mathcal{O}(n)\}$.
- An algebra over an operad is given by a linear space V together with operations from the operads. $\rho : \mathcal{O}(n) \to Hom(V^{\otimes n}, V)$
- The operations $\cup, \{- \bullet -\}$ are such operations. The operad is called e_2.
- The homology of an operad is an operad. In particular $e_2 = H_*(E_2)$, where E_2 are the little two cubes, which is a topological operad. One can also use little discs D_2. The operad structure is plugging discs into discs.
- What about the $\circ_i$. These are chain level operations, as is $\cup$.

Deligne's conjecture

Deligne's conjecture/question

Is there a operad chain model $C_*(D_2)$ with operations induced by the operad structure on D_2 which acts on $CH^*(A, A)$ inducing the operadic Gerstenhaber structure on $HH^*(A, A)$.

Short answer

Yes.

Deligne's conjecture

Some details

Need to fix signs carfully. (Not today)

The statement has several parts

1. A chain lift that provides the operations.
2. An operadic chain lift that provides operadic operations.
3. Showing that the chain model is a chain model for the little discs.
4. Showing that the chain model has an operadic structure induced from that of the little discs.

Levels

Interelations

- Spaces/geometry $\leftrightarrow$ Cell models/chains/dg $\leftrightarrow$ (co)homology.
- weak/dg operations $\leftrightarrow$ algebraic structures
- linear operations $\leftrightarrow$ operadic operations $\leftrightarrow$ operadic operations induces from geometric level.
- dg–operations $\leftrightarrow$ dg–operadic operations$\leftrightarrow$ dg–operadic operations induces from geometric level.
- dg on chain $\leftrightarrow$ compactifications and geometry.

Deligne's conjecture: History

Answers, sometimes almost or partial, good sources of structures

Getzler–Jones '94. (Cell model not operadic)
Tamarkin-Kontsevich 97-98
Voronov 98 (works over $\mathbb{Q}$)
Kontsevich-Soibelman 99 (A_∞, Minimal operad M.)
McClure-Smith '00 (Proof has wrong recognition. Surjection operad)
McClure-Smith '01 (Not really proven to be operadic. New monoidal product $\boxtimes$, new operations $\sqcup, \square_i$)
Berger-Fresse '02 (uses McClure Smith)
Kaufmann '03 (New proof. Polysimplicial cell complex. Uses cacti a new version for D_2. Also fixes recognition in McClure-Smith)
...

More operations

Generalizations

1. Khalkhali '99, operations on cyclic chains.
2. Menichi '03 BV on cohomology. Assume A is Frobenius
3. Kaufmann '04 Proof of cyclic Deligne conjecture (all parts) conjectured by Tamarkin-Tsygan in '00.
4. Tradler-Zeinalian. '04 combinatorial operations on chain level.
5. Tradler-Zeinalian. '06 Algebraic string operations. Includes open/closed operations
6. Kaufmann '06 Package of *operadic* actions from surfaces, fixes second McClure–Smith proof.
7. Kaufmann-Schwell '07 proof of A_∞ using polytopes.
8. Kaufmann '09 Open/closed *operadic* operations
9. Ward '11. Cyclic A_∞.

Even more

More

1. Wahl-Westerland '11. Same closed part as '06, but A_∞, open part (D-brane, vs. open string), saturated for closed surfaces.
2. Wang '16 operations Tate-Hochschild complex using Cacti.
3. Rivera–Wang '17
4. K.–Rivera–Wang, today
5. Nest-Tsygan. Animation and a whole book forthcoming.

Frobenius algebras

Notation

If A is a Frobenius algebra then it is isomorphic as an A-A-bimod to its dual.

- $\langle a, b\rangle, \epsilon(a) = \langle a, 1\rangle, \int_n a_1 \cdots a_n = \epsilon(a_1 \cdots a_n)$
- $\Delta = \mu^\dagger$, $\langle a, bc\rangle = \sum \langle a^{(1)}, b\rangle\langle a^{(2)}, c\rangle$.
- $\Delta(1) = C = \sum C^{(1)} \otimes C^{(2)}$ (Casimir), $\mu\Delta(1) = \sum C^{(1)} C^{(2)} = e$ (Euler class).

Main isomorphisms

- $CH^n(A) \simeq A \otimes \check{A}^{\otimes n} \simeq \check{A}^{\otimes n+1} \simeq A^{\otimes n+1}$
- $CH_n(A) = A^{\otimes n+1}$
- So $CH^* \simeq CH_* \simeq \bar{T}A = A \otimes TA$.
- Moreover the canonical pairing is given by $\langle -, -\rangle^{\otimes n}$ and the differentials are adjoint.

Basic algebraic operations

Monoidal structure $\otimes$ and coproduct

For A-A bimodules consider the two monoidal products
$M \otimes N = M \otimes_k N, \Delta : \bar{T}A \to \bar{T}A \otimes \bar{T}A$

$$\Delta(a_0 \otimes \cdots \otimes a_n) = \sum (a_0 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)$$

The iterates are $\Delta^{[n]} : \bar{T}A \to \bar{T}A^{\otimes n}$

Non–unital monoidal structure $\boxtimes$ and coproduct $\Diamond$

$M \boxtimes N := M \otimes_k A \otimes_k N, \Diamond : TA \to TA \boxtimes TA$

$$\Diamond(a_0 \otimes \cdots \otimes a_n) = \sum (a_0 \otimes \cdots \otimes a_{i-1}) \otimes a_i \otimes (a_{i+1} \otimes \cdots \otimes a_n)$$

The iterates are $\Diamond^{[n]} : TA \to TA^{\boxtimes n}$.

Relations

Inserting units

$id \otimes \eta \otimes id : M \otimes N \simeq M \otimes k \otimes N \to M \otimes A \otimes N = M \boxtimes N.$

We will use:

$\eta_{tot} : A \otimes M_1 \otimes \cdots \otimes M_n \to k \boxtimes M_1 \boxtimes \cdots \boxtimes M_n$ inserts 1 into every slot of A except the first.

Dual using the coaugmentation $\epsilon : A \to k$

$id \otimes \epsilon \otimes id : N \boxtimes M = M \otimes A \otimes N \to M \otimes N$

Algebraic universal operations after [Kau08]

Notation

- Note $\bar{T}A = k \boxtimes TA = A \otimes TA$.
- For $\bar{T}A$ by abuse of notation we write $\lozenge^{[n]}$ for

$$id \otimes \lozenge^{[n]} : k \boxtimes TA \to k \boxtimes TA^{\boxtimes n}$$

  Elements are alternating words in A and $\bar{T}A$:
  $a_0 \otimes w_1 \otimes a_1 \otimes w_2 \cdots a_{n-1} \otimes w_n \in A \otimes \bar{T}A \otimes A \otimes \bar{T}A \otimes \cdots A \otimes \bar{T}A$.
- We will call the factors of $\bar{T}A$ blocks and the factors A interstices and the first factor of A the module variable.

Algebraic universal operations after [Kau08]

Universal formula

$$Y((n_1,\dots,n_n),i) : \bar{T}A^{\otimes n} \to k = \bigotimes_{k,l}\Big(\int_{m_l}\Big)\otimes\langle -,-\rangle_{\bar{T}A}^{\otimes k_j}\circ\sigma\circ\bigotimes_{i=1}^{n}\lozenge^{[n_i]}$$

Where σ is a block permutation permuting the factors of A and $\bar{T}A$. A, $\langle -,-\rangle_{\bar{T}A}$ only pairs two factors of $\bar{T}A$ which are determined by the involution i and the factors in $\int_{m_i}$ are determined by this data as integrals along cycles of a Ribbon graph.

Graphical interpretation

Graph Γ associated to data

The graph will have n vertices. There will be n_i half edges at each vertex. These correspond to the blocks $\bar{T}A$.
The involution i defined edges via its orbits.

Marked ribbon graph structure from data

The half edges have a linear order at each vertex. The associated cyclic order defines a ribbon graph structure.

Angles

A pair of consecutive edges in the associated cyclic order is called *an angle*. These correspond to the interstices A.
The angle between the first and the last half edge is called the module angle.

Illustration

Graphical/Geometric interpretation

Cycles determine the $\int_{m_l}$

Each ribbon graph has cycles given by the involution followed by the successor in the cyclic order.
The integrals are along the angles of the cycles.

Theorem [Kau08]

These are linear operations which define operadic correlation functions for open cells of the moduli space $M_{g,n}$ which form an opread under composition. Using the associated graded of a filtration on the cellular chains, this is a cyclic dg–operadic action for the cyclic operad structure of [KLP03].

Remark

This is actual a modular operad action and extends to an open/closed action [KP06, Kau10].

Generalization

Associated surface with arcs

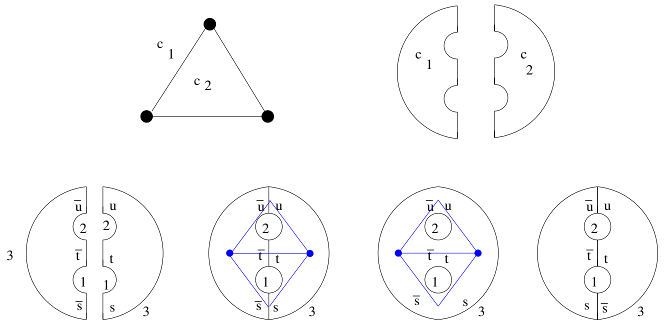
Given such a ribbon graph.

1. Take a 2l–gon for each cycle of length l. Where the sides alternate between angles and edges.
2. Glue together these polygons along the sides in the orbit of the involution.
3. The result is a surface with boundary which is made up out of the angles. There will be n such boundaries.
4. The image of the edges are arcs running from boundary to boundary. These partition each boundary into sectors. There is exactly one sector marked by the module variable in each boundary.

Dual graphs

Proposition [Kau08]

The dual ribbon graph of the surface is also dual to the graph Γ.



Extension to boundary

Kontsevich/Penner compactification

One can extend the correlation functions to the Kontsevich/Penner/combinatorial compactification.
This can be done with polycyclic genus and puncture marked graphs, cf. [BK22].

For Arcs

Let a be a system of arcs and p be a function $a \to \mathbb{N}$.

$$Y(a^p) = \Big(\bigotimes_{S \in Comp(\alpha^p)} Y_A(S) \otimes \bigotimes_{i=1}^{k} \eta^{n_i - 1} \Big) \circ \sigma \circ \bigotimes_{i=0}^{n} \lozenge^{n_i} \quad (3)$$

Here $Y_A(S)$ is the value of the *open TFT* associated to the FA A on S, cf. [Kau18].

Prop type operations

Input/output

The correlation functions $Y : \bar{T}A^{\otimes N}$ are of the form:

$$Y(c)_{p_i,q_j} \in Hom(A^{\otimes p_1+1} \otimes \cdots \otimes A^{\otimes p_n+1} \otimes A^{\otimes q_1+1} \otimes \cdots \otimes A^{\otimes q_m+1}, k)$$

where we partitioned N into $n+m$ and c is a cell/decorated graph. Dualizing the $A^{\otimes q_i+1}$ one obtains a PROP action on $\bar{T}A$:

$$\hat{Y}(c)_{p_i,q_j} \in Hom(A^{\otimes p_1+1} \otimes \cdots \otimes A^{\otimes p_n+1}, A^{\otimes q_1+1} \otimes \cdots \otimes A^{\otimes q_m+1})$$

Finally identifying the $A^{\otimes k+1} \simeq CH^k(A,A)$, one obtains operations

$$op_{CH}(c)_{p_i,q_j} \in \mathrm{Hom}(\bigotimes_{i=1}^{n} CH^{p_i}, \bigotimes_{j=1}^{m} CH^{q_j})$$

Chas–Sullivan string topology type operations

Sullivan space

This spaces retracts to a CW complex. The cells are cells of the Kontsevich/Penner compactification, where the graph allows for a partition into input and output vertices (arcs only from in to out) and all inputs contain arcs and a choice of such an input/output designation.
Note there may be none, one, two or several. If arcs are only from in to out, one switch their roles —a kind of in/out duality. This might violate the input condition.

Action with in/out distinction

Given an input/output distinction the operation is defined by

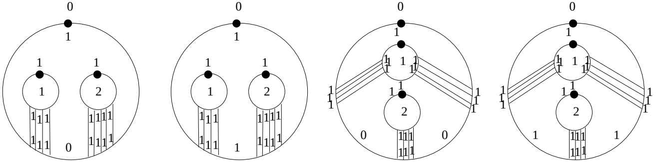
$$\hat{Y}(c, i/o) := \hat{Y}(c)(id^{in} \otimes \eta^{out}_{tot})$$

All angles on the output except the module angle contribute $1 \in A$.

Actions

Theorem [Kau07, Kau08]

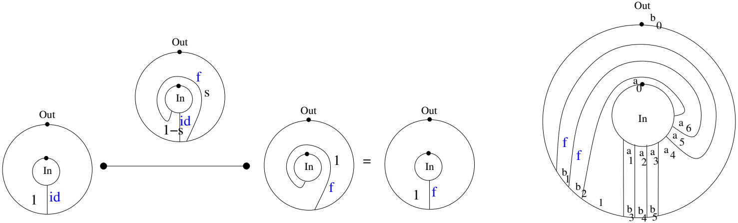
The Sullivan space has a weak prop structure and a chain model that is a dg-Prop.
The operations above give a dg–prop action of this chain level prop on $CH^*(A, A)$ for a Frobenius algebra A which lifts as operadic correlation functions to a weak Frobenius algebra.



Examples of the angle marked partitioned families yielding $\cup$, $\sqcup$, $\circ_i$ and $\square_i$

$$\square_i(f,g)(a_1,\dots,a_{n+m+2}) = \\ f(a_1,\dots,a_{i-1},a_i g(a_{i+1},\dots a_{i_m})a_{i+m+1},a_{i+m+2},\dots,a_{n+m+2}).$$

Animation and BV



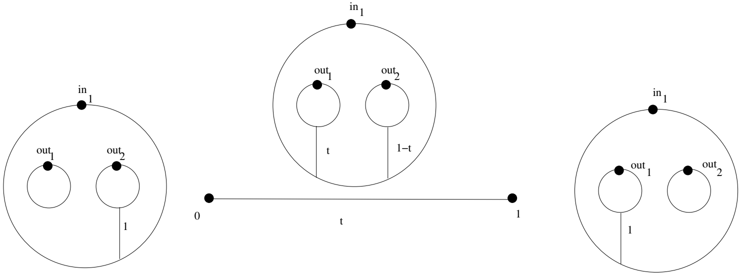
The i/o version gives Connes' operator B. Without inserting 1 this is the operator N. There is the option to insert a function into the factors $\langle -, - \rangle$. This realizes the animation of Nest-Tsygan. The operation on $\bar{T}A$ is

$$a_0 \otimes \cdots \otimes a_n \mapsto \pm \sum \epsilon(a_0)\, a_p \otimes f(a_{p+1}) \otimes \cdots \otimes f(a_n) \otimes a_1 \otimes \cdots \otimes a_{p-1}$$

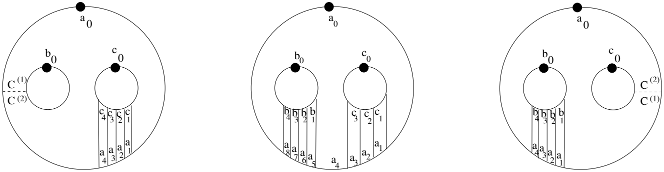
This is a homotopy from $id_{A \otimes TA}$ to $id_A \otimes f^{\otimes}$.

GH coproduct

The 1–dimensional cell for the coproduct and its two boundary points



GH coproduct



Middle: The $(8,3)$–summand of the component of the degree 1 operation on CH^* corresponding to $CH^8 \to CH^3 \otimes CH^4$. Cutting at the arcs yields one octagon P_8 and 7 quadrilaterals (P_4s).

Left: the component of the ∂_0 boundary operation $CH^4 \to CH^4 \otimes CH^0$.

Right: The component of the ∂_1 boundary operation $CH^4 \to CH^0 \otimes CH^4$. The extra cut for the annulus is the dotted line and decorated by $C = \Delta(1) = C^{(1)} \otimes C^{(2)}$.

GH coproduct

Theorem [Kau18]

Given a Frobenius algebra A consider $CH := CH^*(A, A)$. The cell for the coproduct given above acts, as a coproduct morphism

$$\Delta_{CH} \in Hom(CH, CH^{\otimes 2})$$

The formulas for its non–zero components $\Delta_{CH}(f) \in \bigoplus_{p+q=n-1} CH^p \otimes CH^q$ are explicitly given by

$$\begin{aligned} &\Delta_{CH}(f)[(a_1 \otimes \cdots \otimes a_p) \otimes (a_{p+1} \otimes \cdots \otimes a_{n-1})] \\ &= (-1)^p \sum_{C_1, C_2} C_1^{(1)} f(a_1 \otimes \cdots \otimes a_p \otimes C_2^{(1)} C_1^{(2)} \otimes a_{p+1} \otimes \cdots \otimes a_{n-1}) \otimes C_2^{(2)} \end{aligned}$$

GH coproduct

Theorem continued

The boundary of this chain operation is given by the operation of the boundary of the cell. It has two components and the operations corresponding to these are

$$\partial_0\Delta_{CH} : CH^n \to CH^n \otimes CH^0 \text{ and } \partial_1\Delta_{CH} : CH^n \to CH^0 \otimes CH^n$$

which are given by the following explicit formulas: using $CH^0(A, A) = Hom(k, A)$, choosing $a_i \in A$ and $\lambda \in k$

$$\partial_0\Delta_{CH}(f)(\lambda \otimes (d_1 \otimes \cdots \otimes d_n)) = \lambda(1 \otimes f(a_1, \ldots, a_n))\Delta(1)^2$$
$$\partial_1\Delta_{CH}(f)((a_1 \otimes \cdots \otimes a_n) \otimes \lambda) = \lambda\Delta(1)^2(f(a_1, \ldots, a_n) \otimes 1)$$

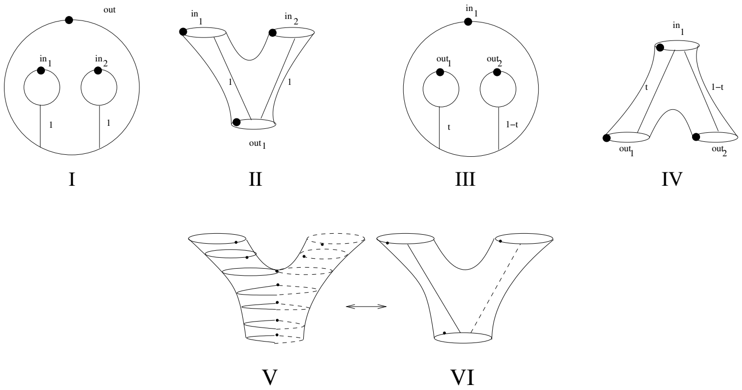
Consequences

Corollary

If A is graded Gorenstein, then the boundary correlation functions vanish unless $a_0, b_0, c_0 \in A_0$.
Dually, $\partial_{0/1}\Delta_{CH}(f) = 0$ unless $f : A^{\otimes n+1} \to A_0 \simeq k$ is a constant map and $\partial_{0/1}\Delta_{CH}(f)$ has image

$$Im(\partial_0 \Delta_{CH}) \subset CH^n(A, A_0) \otimes CH^0(A, A_0) \subset CH^n(A, A) \otimes CH^0(A, A)$$
$$Im(\partial_1 \Delta_{CH}) \subset CH^0(A, A_0) \otimes CH^n(A, A_0) \subset CH^0(A, A) \otimes CH^n(A, A)$$

Duality and Moving strings



Dualizations

Yoga

$Hom(V, W) \simeq W \otimes V^* \simeq (V^* \otimes W)^* \simeq Hom(V \otimes W^*, k) \simeq Hom(W^*, V^*)$

Yoga II

$Hom(\bar{T}A^{n+m}, k) \simeq Hom((CH^*)^{\otimes n}, (CH^*)^{\otimes m}) \simeq CH^{\otimes m} \otimes (CH^{\otimes n})^* \simeq Hom(CH_*^{\otimes m}, CH_*^{\otimes n})$

Mantra

Hochschild chain inputs are Hochschild cochain outputs
Hochschild chain outputs are Hochschild cochain inputs.

Action on Hochschild chains and cochains

ho/co

To implement this, we need to add a new labelling ho/co to the cells/graphs marking the cycles/boundaries. That is each boundary has a double designation in/out and ho/co.

Theorem K-Rivera-Wang

The correlation functions $\hat{Y}$ act on $CH^* \otimes CH_*$ via

$$\hat{Y}(id^{in\,co} \otimes \eta_{tot}^{in\,ho} \otimes \eta_{tot}^{out\,co} \otimes id^{in\,ho})$$

This is a 2–colored dg–prop action extending the operations of [RW19].

GH product

Example

The degree 1, $(2,1)$ product on CH_* given by the formula

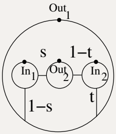
$$b_0 \otimes \cdots \otimes b_p \cup c_0 \otimes \cdots \otimes c_{n-p-1}$$
$$= \pm \sum b_0 C^{(1)} \otimes c_1 \otimes \cdots \otimes c_{n-p-1} \otimes c_0 C^{(2)} \otimes b_1 \otimes \cdots \otimes b_p$$

Double Poisson bracket

Proposition

In particular the mixed triple products of [RW19] correspond to a map $CH^* \otimes CH^* \to CH^* \otimes CH^*$ which is a double Poisson/Gerstenhaber bracket.

Graph and operation



$\{\{\ ,\ \}\}$

$$(a_0 \otimes \cdots \otimes a_n) \otimes (b_0 \otimes \cdots \otimes b_m) \mapsto$$

$$\sum_{p,q} \pm \langle a_p, b_q \rangle (C^{(2)} b_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \otimes b_{q+1} \otimes \cdots \otimes b_m)$$

$$\otimes (C^{(1)} a_0 \otimes b_1 \otimes \cdots \otimes b_{q-1} \otimes a_{p+1} \otimes \cdots \otimes a_n)$$

Double Poisson

Comments

1. This is independent, but related to Iyudu-Kontsevich-Vlassopoulos.
2. This is part of a series of even higher brackets.

Action on TH

Theorem K.-Rivera-Wang

- There is a differential on a two colored version of the Sullivan space making it into a CW complex. The cells have a a dg action on the Tate-Hochschild complex $(\mathcal{D}^*(A,A),\delta) = \mathrm{Cone}(\gamma).\ CH_*(A,A)[1] \oplus CH^*(A,A)$ $\gamma(a) = C^{(1)}aC^{(2)} : A = CH_0(A) \to A = CH^0(A)$.
- The operations on the Tate-Hochschild complex given by this dg action coincide with the linear operations after normalizing the underlying PROP.
- The new action may be obtained via a cellular chain complex of a new compactification of the Sullivan space, which can be identified with a cover of the Kontsevich–Penner compactification.

The End

Thank you!

Extra Page

Extra Page

Extra Page

Clemens Berger and Ralph M. Kaufmann.
Trees, graphs and aggregates: a categorical perspective on combinatorial surface topology, geometry, and algebra.
ArXiv 2201.10537, 2022.

Murray Gerstenhaber.
The cohomology structure of an associative ring.
Ann. of Math. (2), 78:267–288, 1963.

Ralph M. Kaufmann.
Moduli space actions on the Hochschild co-chains of a Frobenius algebra. I. Cell operads.
J. Noncommut. Geom., 1(3):333–384, 2007.

Ralph M. Kaufmann.
Moduli space actions on the Hochschild co-chains of a Frobenius algebra. II. Correlators.
J. Noncommut. Geom., 2(3):283–332, 2008.

Ralph M. Kaufmann.

Open/closed string topology and moduli space actions via open/closed Hochschild actions.
SIGMA Symmetry Integrability Geom. Methods Appl., 6:Paper 036, 33, 2010.

Ralph M. Kaufmann.
A detailed look on actions on hochschild complexes especially the degree 1 co-product and actions on loop spaces.
arXiv:1807.10534, 2018.

Ralph M. Kaufmann, Muriel Livernet, and R. C. Penner.
Arc operads and arc algebras.
Geom. Topol., 7:511–568 (electronic), 2003.

Ralph M. Kaufmann and R. C. Penner.
Closed/open string diagrammatics.
Nuclear Phys. B, 748(3):335–379, 2006.

Manuel Rivera and Zhengfang Wang.
Singular Hochschild cohomology and algebraic string operations.

J. Noncommut. Geom., 13(1):297–361, 2019.