# Regular Ideals and Regular Inclusions

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Graph algebras are $C^*$-algebras defined from directed graphs.

Let $E = (E^0, E^1, r, s)$ be a directed graph.

$E^0$ is the set of vertices, $E^1$ is the set of edges, and $s, r : E^1 \to E^0$ are the range and source maps.

We will assume $E$ has no sources or infinite receivers: that is, for every $v \in E^0$, $0 < |r^{-1}(v)| < \infty$.

**Features of a directed graph.**

A *return path* is a path $e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_1)$ iff $i = n$.

A *cycle* is a return path s.t. for all $j, i$, $s(e_i) = s(e_j)$ only when $j = i$.

A *cycle without entry* is a cycle $e_1 e_2 \cdots e_n$ such that for any $e \in E^0$, $r(e) = r(e_i)$ only if $e = e_i$.

A graph satisfies

- Condition (L) if it has no cycles without entry, and
- Condition (K) if no vertex is the source of exactly one return path.

Exercise: (K) $\Rightarrow$ (L) but (L) $\not\Rightarrow$ (K).

A *Cuntz-Krieger* $E$*-system* associates the elements $v \in E^0$ to orthogonal projections $p_v$ and $e \in E^1$ to partial isometries $s_e$ on a Hilbert space $H$, in a way such that for all $e \in E^1, v \in E^0$,

$$s_e^* s_e = p_{s(e)} \quad \text{and} \quad \sum_{r(e)=v} s_e s_e^* = s_v.$$

$C^*(E)$ is the $C^*$-algebra generated by a universal C-K system.

There is a homomorphism $z \mapsto \gamma_z$ from $\mathbb{T}$ to $\mathrm{Aut}(C^*(E))$ such that for any fixed $a \in C^*(E)$, the map $z \mapsto \gamma_z(a)$ is continuous. This map, given by $\gamma_z(p_v) = p_v$, $\gamma_z(s_e) = zs_e$, is known as the *gauge action*.

For $H \subseteq E^0$ a set of vertices we say

- $H$ is *hereditary* if for all $e \in E^1$, whenever $r(e) \in H$ then $s(e) \in H$,
- $H$ is *saturated* if for all $v \in E^0$, $s(r^{-1}(v)) \subseteq H$ implies $v \in H$,

and we denote by $I(H)$ the ideal in $C^*(E)$ generated by $\{p_v, v \in H\}$.

For $J \subseteq C^*(E)$ is a (closed, two-sided) ideal, denote

$$H(J) = \{v \in E^0 \,|\, p_v \in J\}.$$

Facts:

1. For any ideal $J$, $H(J)$ is saturated and hereditary.
2. For any $H \subseteq E^0$, $H(I(H))$ is the smallest saturated, hereditary set containing $H$.
3. For any ideal $J \subseteq C^*(E)$, $I(H(J))$ is the largest gauge-invariant ideal contained in $J$. In particular, $I(H(J)) = \bigcap_{z \in \mathbb{T}} \gamma_z(J)$.
4. There is a one-to-one correspondence between gauge-invariant ideals of $C^*(E)$ and saturated hereditary subsets of $E^0$.

If $J \subseteq C^*(E)$ is a closed ideal then denote by $E/J$ the largest subgraph of $E$ with no vertex in $H(J)$:

$$(E/J)^0 = E^0 \setminus H(J); \quad (E/J)^1 = E^1 \setminus s^{-1}(H(J))$$

and range and source maps inherited from $E$.

**Theorem** (Bates-Pask-Raeburn-Szymański, 2000)
If $J$ is a gauge-invariant ideal of $C^*(E)$, then $C^*(E)/J \cong C^*(E/J)$.

Moreover, if $E$ satisfies Condition (K), then the graph $F = E/J$ also satisfies Condition (K).

The analogous statement does not hold for Condition (L).

Let $A$ be a $C^*$-algebra. For a subset $X \subseteq A$, let

$$X^{\perp} = \{a \in A \,|\, \text{for all } x \in X,\ xa = ax = 0\}.$$

We call an ideal $J \subseteq A$ *regular* if $J = J^{\perp\perp}$.

**Theorem** (Brown-Fuller-Pitts-R, 2019) If $E$ satisfies (L) and $J$ is a regular gauge-invariant ideal of $C^*(E)$, then $E/J$ satisfies (L).

Ingredients of the proof:

**Lemma:** For $w \in E^0$, let $T(w) = \{s(\alpha) \,|\, \alpha \in E^*, r(\alpha) = w\}$.
Then a gauge-invariant ideal $J$ is regular iff

$$w \in H(J) \;\Leftrightarrow\; \forall v \in T(w)\; \exists u \in T(v) \cap H(J).$$

**Proof of Theorem**

Let $\lambda = e_1 e_2 \cdots e_n$ be a cycle in $F$. Since $E$ has Condition (L), $\mathcal{E} := \{e \in E^1 : e$ is an entrance for $\lambda\} \neq \emptyset$. If $\lambda$ has no entry in $E/J$ then $s(\mathcal{E}) \subseteq H(J)$. Since $J$ is regular, it follows from the lemma that all vertices of $\lambda$ are in $H(J)$, contradicting that $\lambda$ is a cycle in $E/J$. Hence $F$ has Condition (L).

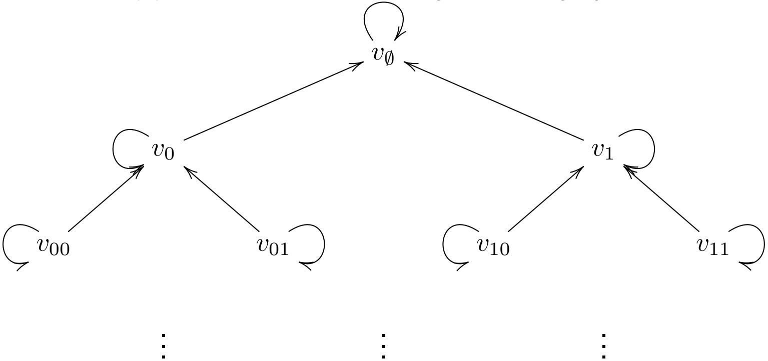
**Theorem** ([BFPR1] If $E$ satisfies (L) and $J$ is a regular gauge-invariant ideal of $C^*(E)$, then $E/J$ satisfies (L).

**Corollary** If $E$ satisfies (L) then every regular ideal of $C^*(E)$ is gauge-invariant.

Sketch of proof of corollary:

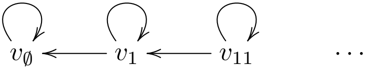
- If $J$ is regular then $I(H(J))$ is regular and gauge-invariant.
- Thus $E/I(H(J))$ satisfies Condition (L) by the theorem, and thus so does $E/J$.
- Fact: For any ideal $J$, if $E/J$ satisfies Condition (L) then $J$ is gauge-invariant. (Uses C-K Uniqueness Thm)

There are graphs $E$ with non-regular ideals $J$ in $C^*(E)$ such that $E/J$ satisfies (L). Consider the following directed graph $E$:




Let $H = T(v_0) \sqcup \left( \bigsqcup_{i\in\mathbb{N}^+} T(v_{\underbrace{11\ldots1}_{i}0}) \right)$.

Then $H$ is a saturated hereditary set.
$E/I(H) = (E^0 \backslash H, s^{-1}(E^0 \backslash H), r, s)$ is the graph




which has Condition $(L)$.

However, $I(H)$ is not regular because for example
$w = v_0 \notin H = H(I(H))$ though $\forall v \in T(w)\ \exists u \in T(v) \cap H$.

If $B \subseteq A$ contains an approximate unit for $A$ and $A$ is the closure of the span of the normalizers of $B$, then we say the inclusion $B \subset A$ is *regular*.

**Theorem** ([BFPR2])

Let $B \subseteq A$ be a regular inclusion of $C^*$-algebras satisfying the regular ideal intersection property. Further assume there is an invariant faithful conditional expectation $E\colon A \to B$. Then

(a) the invariant regular ideals of $B$ form a complete Boolean algebra.

(b) There is a Boolean algebra isomorphism between the regular ideals of $A$ and the invariant regular ideals of $B$ given by the maps $J \mapsto J \cap B$ and $K \mapsto J_K := \{a \in A\colon E(a^*a) \in \iota(K)\}$.

Notes: If $K \subset B$ is an invariant ideal then $J_K \cap B = K$.
If $K$ is regular and invariant then also $J_K$ is regular and $J_K \cap B = K = E(J_K)$.

**Theorem** ([BFPR2]) Suppose $B \subseteq A$ is a regular inclusion, $B$ has the intersection property in $A$, and $E\colon A \to B$ is an invariant faithful conditional expectation. Let $J \trianglelefteq A$ be a regular ideal. Then $B/(J \cap B)$ has the intersection property in $A/J$.

**Theorem** ([BFPR2]) If $D$ is a Cartan subalgebra of a $C^*$-algebra $A$ and $J \trianglelefteq A$ is a regular ideal, then $D/(J \cap D)$ is a Cartan subalgebra of $A/J$.

Final comments:

- Much of this has been reproduced/adapted to $k$-graphs by Tim Schenkel.
- Several results in [BFPR1] can be proved in alternative ways using the results of [BFPR2]
- Results in [BFPR2] stated above requiring conditional expectations are also adapted to the setting where only a pseudo-expectation is available.
- Thank you for attending today!

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