Graded and filtered K-theories for Leavitt path algebras

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- A short exact sequence of Grothendieck groups for LPAs
- Filtered K-theory for LPAs
- Main result

Leavitt algebra

The Leavitt algebra is the free associative algebra A over a field k generated by $\{x_i, y_i \mid 1 \leq i \leq n\}$ subject to the relations

$$x_i y_j = \delta_{ij}, \forall 1 \leq i, j \leq n, \text{ and } \sum_{i=1}^{n} y_i x_i = 1.$$

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$$\begin{aligned} \phi : A &\longrightarrow A^n \\ a &\longmapsto (x_1 a, x_2 a, \cdots, x_n a) \end{aligned}$$

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$$\begin{aligned} \psi : A^n &\longrightarrow A \\ (a_1, \cdots, a_n) &\longmapsto y_1 a_1 + \cdots + y_n a_n. \end{aligned}$$

W. G. Leavitt, Trans. Amer. Math. Soc. 1962

Leavitt path algebra

Algebra	C^***-algebra**
Leavitt algebras, 1962	Cuntz C^*-algebras, 1977
Leavitt path algebras, 2005	Cuntz–Krieger C^*-algebras, 1980 graph C^*-algebras, 1997

- J. Cuntz, Commun. Math. Phys.,1977.
- J. Cuntz, W. Krieger, Invent. Math., 1980.
- P. Ara, M. A. Moreno, E. Pardo, Algebr. Represent. Theory, 2007
- G. Abrams, G. Aranda Pino, J. Algebra, 2005.

Leavitt path algebras

$$\mathcal{L} : \text{Graphs} \longrightarrow \text{Algebras}$$

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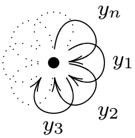

$$\longrightarrow \mathcal{L}(1, n)$$

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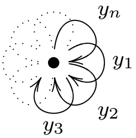

$$\longrightarrow \mathcal{L}(1, n)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix} = I_n$$

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Definition for Lpa

Definition

For a graph E and a field k, we define the *Leavitt path algebra of* E over k, denoted by $L_k(E)$ (or simply denoted by $L(E)$), to be the free associative k-algebra generated by the sets $\{v \mid v \in E^0\}$, $\{\alpha \mid \alpha \in E^1\}$ and $\{\alpha^* \mid \alpha \in E^1\}$, subject to the relations

1. $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$;
2. $s(\alpha)\alpha = \alpha r(\alpha) = \alpha$ and $r(\alpha)\alpha^* = \alpha^* s(\alpha) = \alpha^*$ for all $\alpha \in E^1$;
3. $\alpha^*\alpha' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^1$;
4. $\sum_{\{\alpha \in E^1, s(\alpha)=v\}} \alpha\alpha^* = v$ for every $v \in E^0$ for which $s^{-1}(v)$ is nonempty and a finite set.

Definition for Lpa

$L(E)$ is a naturally $\mathbb{Z}$-graded algebra by lengths of paths:

$$L(E) = \oplus_{n \in \mathbb{Z}} L(E)^n$$

s.t.

$$L(E)^n L(E)^m \subseteq L(E)^{n+m}.$$

Examples

Example

E_1 : 

Examples

Example

E_1 : 

The Leavitt path algebra $L(E_1)$ is the Laurent polynomial ring $k[x, x^{-1}]$.

Examples

Example

$$E_2 : \qquad \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$$

Examples

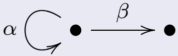
Example

$$E_2 : \qquad \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$$

$$\begin{aligned} f : L(E_2) &\longrightarrow M_3(k) \\ v_i &\longmapsto e_{ii}, \\ \alpha &\longmapsto e_{12}, \\ \beta &\longmapsto e_{23}, \\ \alpha^* &\longmapsto e_{21}, \\ \beta^* &\longmapsto e_{32}. \end{aligned}$$

Examples

Example

$E_3 :$ 

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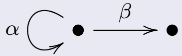

$$\begin{aligned} f : k < x, y > &\longrightarrow L(E_3) \\ 1 &\longmapsto v_1 + v_2, \\ x &\longmapsto \alpha + \beta, \\ y &\longmapsto \alpha^* + \beta^*. \end{aligned}$$

Examples

Example

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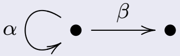
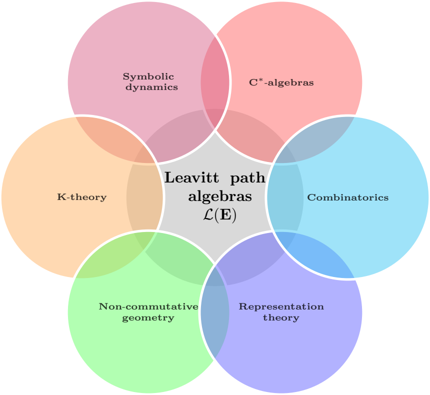


$$\begin{aligned} f : k < x, y > &\longrightarrow L(E_3) \\ 1 &\longmapsto v_1 + v_2, \\ x &\longmapsto \alpha + \beta, \\ y &\longmapsto \alpha^* + \beta^*. \end{aligned}$$

The Leavitt path algebra $L(E_3) \cong k < x, y > /(xy - 1)$.

Related research areas



The stable derived category

- the stable derived category of an algebra is compactly generated

H. Krause, Compos. Math., 2005.

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- describe the stable derived category for $kE/kE^{\geq 2}$ in terms of derived category of Leavitt path algebra

S. Paul Smith, Adv. Math. 2012.

X.W. Chen, D. Yang, Int. Math. Res. Not., 2015.

An explicit generator

Theorem

Let $A = kE/kE^{\geq 2}$.

(1) *The injective Leavitt complex* $\mathcal{I}^{\bullet}$ *of* E *is a compact generator for the stable derived category.*

(2) *The dg endomorphism algebra* $\mathrm{End}_A(\mathcal{I}^{\bullet})$ *is quasi-isomorphic to the Leavitt path algebra* $L_k(E)$.

H. Li, Algebr. Represent. Theor., 2018.

H. Li, P. Edinburgh Math. Soc., 2018.

The graph monoid

Ara, Moreno and Pardo introduced the graph monoid M_E for a row-finite graph E:

$$M_E = \langle v \in E^0 \,|\, v = \sum_{v \to u} u \rangle.$$

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They showed that

$$\mathcal{V}(L(E)) \cong M_E,$$

where $\mathcal{V}(A)$ is the monoid of isomorphism classes of f.g. projective modules over A.

The graph monoid

Ara and Goodearl gave the definition of M_E when E is not necessarily row-finite:

The generators $v \in E^0$ of the abelian monoid M_E for E are supplemented by generators q_Z as Z runs through all nonempty finite subsets of $s^{-1}(v)$ for infinite emitters v. The relations are

(1) $v = \sum_{e \in s^{-1}(v)} r(e)$ for all regular vertices $v \in E^0$;

(2) $v = \sum_{e \in Z} r(e) + q_Z$ for all infinite emitters $v \in E^0$ and all

(3) $q_{Z_1} = \sum_{e \in Z_2 \setminus Z_1} r(e) + q_{Z_2}$ for all nonempty finite sets $Z_1 \subseteq Z_2 \subseteq s^{-1}(v)$, where $v \in E^0$ is an infinite emitter.

The graded graph monoid

The "graded" graph monoid of a row-finite graph is M_E^{gr}, the abelian/commutative monoid generated by $\{v(i) \mid v \in E^0, i \in \mathbb{Z}\}$ subject to

$$v(i) = \sum_{v \to u} u(i+1),$$

for every $v \in E^0$ that there exist edges starting at v and $i \in \mathbb{Z}$.
We have the definition of the graded graph monoid for an arbitrary graph.

P. Ara, R. Hazrat, H. Li, A. Sims, Algebra and Number Theory, 2018.

Group completion

The group completion of a commutative monoid V (also called the Grothendieck group of V):

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The group completion of a commutative monoid V (also called the Grothendieck group of V):
A relation $\sim$ on $V \times V$: $(x_1, y_1) \sim (x_2, y_2)$ if there is a $z \in V$ s.t.

$$x_1 + y_2 + z = y_1 + x_2 + z.$$

We define the equivalence classes of $V \times V / \sim$ by

$$V^+ = \{[x, y] | (x, y) \in V \times V\}$$

and

$$[(x_1, y_1)] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2].$$

K-theory

$K_0^{\mathrm{gr}}(L(E))$: the graded Grothendieck group of $L(E)$ which is the group completion of M_E^{gr}

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$$n[P] = [P(n)]$$

with $P(n)_m = P_{n+m}$.

This makes $K_0^{\mathrm{gr}}(L(E))$ a $\mathbb{Z}[x, x^{-1}]$-module.

K-theory

Conjecture: Let E and F be finite graphs, and k a field. Then there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism

$$\phi : K_0^{\mathrm{gr}}(L(E)) \to K_0^{\mathrm{gr}}(L(F))$$

if and only if $L(E)$ is graded Morita equivalent to $L(F)$. Furthermore, if $\phi([L(E)]) = [L(F)]$, then $L(E) \cong_{\mathrm{gr}} L(F)$.

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R. Hazrat, Math. Annalen, 2013.

- Hazrat proved his conjecture for all acyclic, comet, or more generally, polycephalic graphs.
- the conjecture is still open.

K-theory

Theorem: Let E and F be finite graphs without sinks. Then the following statements are equivalent:

(i) $L(E)$ and $L(F)$ are graded Morita equivalent;

(ii) $L(E)$ and $L(F)$ are derived equivalent;

(iii) The stable derived categories for $kE/kE^{\geq 2}$ and $kF/kF^{\geq 2}$ are triangulated equivalent;

(iv) $\mathcal{D}_{\text{sg}}(kE/kE^{\geq 2})$ and $\mathcal{D}_{\text{sg}}(kF/kF^{\geq 2})$ are singular equivalent.

X.W.Chen, D. Yang, International Mathematics Research Notices, 2015.

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X.W.Chen, D. Yang, International Mathematics Research Notices, 2015.

If $L(E)$ and $L(F)$ are graded Morita equivalent, then there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism

$$K_0^{\text{gr}}(L(E)) \cong K_0^{\text{gr}}(L(F)).$$

$\mathcal{V}^{\mathrm{gr}}(L(E))$

Theorem

(1) $M_E^{\mathrm{gr}} \cong \mathcal{V}^{\mathrm{gr}}(L(E))$. Here $\mathcal{V}^{\mathrm{gr}}(L(E))$ is the monoid of graded-isomorphism classes of graded f.g. projective $L(E)$-modules.

(2) $\mathcal{V}^{\mathrm{gr}}(L(E))$ is cancellative and $\mathcal{V}^{\mathrm{gr}}(L(E)) \to K_0^{\mathrm{gr}}(L(E))$ is injective.

(3) Then there is a one-to-one correspondence between $\mathcal{L}^{\mathrm{gr}}\big(L(E)\big)$ the lattice of graded two-sided ideals of $L(E)$ and the lattice of the graded ordered ideals of $K_0^{\mathrm{gr}}(L(E))$.

P. Ara, R. Hazrat, H. Li, A. Sims, Algebra and Number Theory, 2018.

K_0 and K_1 groups for LPAs

The (algebraic) K_0 and K_1 groups of $L(E)$ was computed by Ara, Brustenga, and Cortiñas [1], and Gabe, Ruiz, Tomforde and Whalen [2] extended to Leavitt path algebras of countable graphs with infinite emitters.

1. P. Ara, M. Brustenga, G. Cortiñas, Münster J. Math., 2009.

2. J. Gabe, E. Ruiz, M. Tomforde, T. Whalen, J. Algebra, 2015.

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2. J. Gabe, E. Ruiz, M. Tomforde, T. Whalen, J. Algebra, 2015.

We talk about finite graphs here. Decompose the vertices of a finite graph E as $E^0 = R \sqcup S$, where S is the set of sinks and $R = E^0 \setminus S$. With respect to this decomposition we write the adjacency matrix of E as

$$A_E = \begin{pmatrix} B_E & C_E \\ 0 & 0 \end{pmatrix}.$$

Here, we list the vertices in R first and then the vertices in S.

K_0 and K_1 groups for LPAs

More precisely, they have the computation of K-theory in the following.

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Lemma

Let E be a finite graph and $L(E)$ the Leavitt path algebra over a field k. Denote by $k^\times = k \setminus \{0\}$ the multiplicative group of the units of k. We have

(i) $K_0(L(E)) \cong \mathrm{Coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}^R \longrightarrow \mathbb{Z}^{E^0}\right)$;

(ii) $K_1(L(E))$ *is isomorphic to a direct sum:*

$$\mathrm{Coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : (k^\times)^R \longrightarrow (k^\times)^{E^0}\right) \bigoplus \mathrm{Ker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}^R \longrightarrow \mathbb{Z}^{E^0}\right)$$

Graded prime ideals

A graded ideal P of $L(E)$ will be a prime ideal if given any graded ideals A, B of $L(E)$, $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$.

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$\mathrm{Spec}^{\mathrm{gr}}(L(E))$: the set of graded prime ideas of $L(E)$, having the *Jacobson topology* with the open sets

$$W(I) = \{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{gr}}(L(E)) \mid I \nsubseteq \mathfrak{p}\}$$

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for I a graded ideal of $L(E)$.

We observe that the lattice of graded (prime) ideals of $L(E)$ is isomorphic to the lattice of order (prime) ideals of $\mathcal{V}^{\mathrm{gr}}(L(E))$.

Lifting

This allows us to lift an order-preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism between the graded Grothendieck groups of two Leavitt path algebras to a natural homeomorphism between the space of spectrums of their graded prime ideals.

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Lemma

Let E, F be finite graphs. Suppose that there exists an order-preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $\varphi : K_0^{\mathrm{gr}}(L(E)) \longrightarrow K_0^{\mathrm{gr}}(L(F))$. Then there exists a homeomorphism of topological spaces

$$\varphi : \mathrm{Spec}^{\mathrm{gr}}(L(E)) \to \mathrm{Spec}^{\mathrm{gr}}(L(F)).$$

An exact sequence for Grothendieck groups for LPAs

Proposition

Let E be a finite (or row-finite) graph. We have the following short exact sequence

$$K_1(L(E)) \xrightarrow{T} K_0^{\mathrm{gr}}(L(E)) \xrightarrow{\phi} K_0^{\mathrm{gr}}(L(E)) \xrightarrow{U} K_0(L(E)) \longrightarrow 0,$$

where ϕ is the homomorphism $\phi(v(i)) = v(i+1) - v(i)$ for $v \in E^0, i \in \mathbb{Z}$, U is the homomorphism such that $U(v(i)) = v$ by forgetting the grading for $v \in E^0, i \in \mathbb{Z}$, and T is the homomorphism given by

$$K_1(L(E)) \cong \operatorname{Coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : (k^\times)^R \longrightarrow (k^\times)^{E^0}\right) \oplus \operatorname{Ker}\phi \xrightarrow{(0 \quad 1)} K_0^{\mathrm{gr}}(L(E))$$

Filtered K-theory

Filtered K-theory: further developed by Eilers, Restorff, Ruiz and Sørensen

Filtered K-theory

Filtered K-theory: further developed by Eilers, Restorff, Ruiz and Sørensen

The sublattice of gauge invariant prime ideals and their subquotient K-groups for C*-algebras can be used as an invariant.

1. S. Eilers, G. Restorff, E. Ruiz, A.P.W. Sørensen, MATRIX Annals, MATRIX Book Series 1, 2016.
2. S. Eilers, G. Restorff, E. Ruiz, A.P.W. Sørensen, Canadian Journal of Mathematics, 2018.

Filtered K-theory

Eilers, Restorff, Ruiz and Sørensen introduced the filtered K-theory in the purely algebraic setting.

Definition

For $k \leq m$, the *filtered K-theory* $\mathrm{FK}_{k,m}(R)$ of a graded ring R is the collection of algebraic K-groups

$$\{K_n(J), K_n(J/I)\}_{k \leq n \leq m}$$

for graded ideas $I \subseteq J$, together with exact sequences

$$K_n(J/I) \xrightarrow{\iota_*} K_n(P/I) \xrightarrow{\pi_*} K_n(P/J) \xrightarrow{\partial_*} K_{n-1}(J/I).$$

for graded ideas $I \subseteq J \subseteq P$ of R and $k \leq n \leq m$.

S. Eilers, G. Restorff, E. Ruiz, A.P.W. Sørensen, MATRIX Annals, MATRIX Book Series 1, 2016.

Filtered K-theory

Theorem

Let E and F be graphs. If there is an isomorphism

$$\mathrm{FK}_{0,1}(L_{\mathbb{C}}(E)) \cong \mathrm{FK}_{0,1}(L_{\mathbb{C}}(F)),$$

then there is an isomorphism

$$\mathrm{FK}_{0,1}^{\mathrm{top}}(C^*(E)) \cong \mathrm{FK}_{0,1}^{\mathrm{top}}(C^*(F)).$$

S. Eilers, G. Restorff, E. Ruiz, A.P.W. Sørensen, MATRIX Annals, MATRIX Book Series 1, 2016.

Not a group homomorphism

Carlsen, Eilers, and Tomforde described a group isomorphism from $\mathrm{Ker}\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}$ to $K_1^{\mathrm{top}}(C^*(E))$.

T. M. Carlsen, S. Eilers, M. Tomforde, J. K-theory, 2012.

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Now we consider the Leavitt path algebra and adapted the above formula, then we have the map

$$\chi_1' : \mathrm{Ker}\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \longrightarrow \mathrm{K}_1(\mathrm{L}(\mathrm{E})).$$

However it is not a group homomorphism.

Reduced K-group

By the long exact sequence of K-groups for Leavitt path algebras, we have

$$k^{\times R} \xrightarrow{\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}} k^{\times E^0} \xrightarrow{\lambda'} K_1(L(E)) \xrightarrow{\xi'} \mathbb{Z}^R \xrightarrow{\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}} \mathbb{Z}^{E^0}$$

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We define G_E to be the subgroup of $K_1(L(E))$ generated by $[-v]_1$ for $v \in E^0$, where $[-v]_1$ is the image of the element $(1, \cdots, -1, \cdots, 1)^t \in k^{\times E^0}$ under the map λ'. Here -1 corresponds to the component of v and 1 corresponds to all the other components.

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Define

$$\overline{K}_1(L(E)) := K_1(L(E))/G_E.$$

We refer to $\overline{K}_1(L(E))$ as the *reduced* K_1*-group* of $L(E)$.

Reduced K-group

Then we have the following commutative diagram with p the natural projection.

$$\begin{array}{ccc} K_1\big(L(E)\big) & \xrightarrow{\quad\quad\xi'\quad\quad} & \operatorname{Ker}\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \\ & {\scriptstyle p}\searrow \qquad \nearrow{\scriptstyle \xi} & \\ & \overline{K}_1\big(L(E)\big) & \end{array}$$

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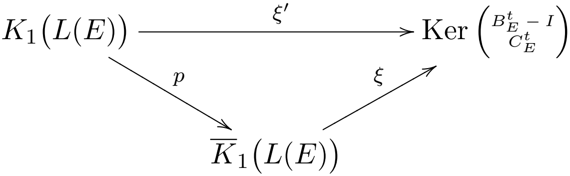

Proposition

The map $\chi_1 : \mathrm{Ker}\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \to \overline{K}_1(L(E))$ *given by* $\chi_1 = p \circ \chi_1'$ *is a group homomorphism and it is a section of* ξ*.*

Quotient of filtered K-theory

Definition

Let E be a row-finite graph. Define the *algebraic filtered* $\overline{K}$*-theory* $\mathrm{F\overline{K}_{0,1}(L(E))}$ as the collection

$$\{\overline{K}_n(J/I)\}_{0\leq n\leq 1}$$

where (I, J) ranges over all the graded ideals of $L(E)$ with $I \subseteq J$,

$$\overline{K}_0(J/I) = K_0(J/I)$$

and $\overline{K}_1(J/I)$ is defined using the fact that J/I is canonically a Leavitt path algebra of a certain graph.

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For $I\subseteq J\subseteq P$, there are natural maps

$$\overline{K}_1(J/I)\to\overline{K}_1(P/I)\to\overline{K}_1(P/J)\to K_0(J/I)\to K_0(P/I)\to K_0(P/J)$$

induced by the long exact sequence of algebraic K-theory.

Main result

Ara, Hazrat, and Li obtained the following result:

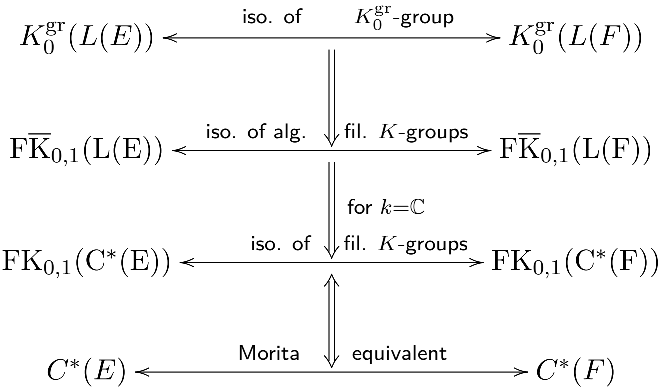
Theorem

Let E, F be row-finite graphs and k a field. Suppose that there exists an order-preserving $\mathbb{Z}[x, x^{-1}]$-isomorphism

$$\varphi : K_0^{\mathrm{gr}}(L_k(E)) \longrightarrow K_0^{\mathrm{gr}}(L_k(F)).$$

Then we have an isomorphism $\mathrm{F}\overline{\mathrm{K}}_{0,1}(L_k(E)) \cong \mathrm{F}\overline{\mathrm{K}}_{0,1}(L_k(F))$.

Relation with C^*-algebras



End

Thank you very much!