Local theta correspondence via C^*-algebras of groups

(joint with Bram Mesland)

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Outline

1. Local theta correspondence
2. Results
3. Some applications

Local Theta Correspondence

Set

- F : a local field of char. 0,
- $(W, \langle \cdot, \cdot \rangle)$: a symplectic vector space over F,

Recall that the **Heisenberg group** $Heis(W)$ is

$$W \oplus F$$

equipped with the rule

$$(w, t) \cdot (w', t') := (w + w', t + t' + \frac{1}{2} \langle w, w' \rangle).$$

Its centre is

$$\mathcal{Z}(Heis(W)) = \{(0, t)\} \simeq F.$$

Local Theta Correspondence

We have

Theorem (Stone-von Neumann)

For each non-trivial character $\chi : F \rightarrow \mathbb{C}^{\times}$, there is a unique equivalence class of irreducible unitary representations of Heis(W) with central character χ.

This representation is known as the **Heisenberg representation**.

Local Theta Correspondence

Its best known model is the so-called **Schrödinger model**: Write

$$W = U \oplus U^*$$

where U, U^* are totally isotropic subspaces. Then

$$\rho : Heis(W) \to \mathrm{GL}(L^2(U))$$

via

$$\rho(u, 0, 0)(f)(x) := f(x + u),$$

$$\rho(0, u^*, 0)(f)(x) := \chi(\langle x, u^* \rangle) f(x),$$

$$\rho(0, 0, t)(f)(x) := \chi(t) f(x),$$

for $f(x) \in L^2(U)$ and $(u, u^*, t) \in Heis(W) = U \oplus U^* \oplus F$.
Note that the space of smooth vectors is the Schwartz space $\mathcal{S}(U)$.

Local Theta Correspondence

We now bring in the automorphisms of $Heis(W)$. Consider

- $\mathrm{Sp}(W) = \{g \in \mathrm{GL}(W) \mid \langle gw, gw' \rangle = \langle w, w' \rangle \ \forall w, w' \in W\}$

Given $g \in \mathrm{Sp}(W)$ and a Heisenberg representation $\rho = \rho_\chi$, the recipe

$$\rho^g((w,t)) := \rho((gw,t))$$

gives us another irreducible unitary representation of $Heis(W)$ with central character χ. Therefore, ρ^g and ρ are equivalent by the Stone-von Neumann theorem: there is an operator M_g such that

$$\rho^g = M_g \circ \rho \circ M_g^{-1}$$

Local Theta Correspondence

There is a unitary operator M_g such that

$$\rho^g = M_g \circ \rho \circ M_g^{-1}.$$

The operator M_g is unique up to a scalar. Thus the map

$$\omega : g \to [M_g]$$

gives a projective unitary representation of $\mathrm{Sp}(W)$, called the **oscillator representation**.

Note that dependence of ω on our fixed character χ is rather weak.

Local Theta Correspondence

Theorem (Segal-Shale-Weil)

The oscillator representation ω *lifts to a true representation of the metaplectic cover* $\mathrm{Mp}(W)$

$$1 \to \{\pm 1\} \to \mathrm{Mp}(W) \to \mathrm{Sp}(W) \to 1$$

of $\mathrm{Sp}(W)$*.*

The metaplectic group is not linear and ω is a *minimal* representation of $\mathrm{Mp}(W)$.

Local Theta Correspondence

Definition

A pair (G, H) of subgroups of $\mathrm{Sp}(W)$ is called a **dual pair** if

- G, H act completely reducibly on W,
- G and H are each others' centralizers.

Example ("ortho-symplectic pair")

$(\mathrm{Sp}(V'), O(V))$ form a dual pair in $\mathrm{Sp}(V' \otimes V)$.

There are also unitary pairs, quaternionic pairs and general linear pairs.

Local Theta Correspondence

Kudla has shown that (almost!) all dual pairs admit a splitting

$$\begin{array}{ccc} & & \mathrm{Mp}(V' \otimes V) \\ & \nearrow & \downarrow \\ G \times H & \longrightarrow & \mathrm{Sp}(V' \otimes V) \end{array}$$

so that we can pullback the oscillator representation of $\mathrm{Mp}(V' \otimes V)$ to $G \times H$:

$$\omega : G \times H \longrightarrow \mathcal{U}(\mathcal{H}).$$

Local Theta Correspondence

In many cases, Howe has shown that

Theorem (Double Commutant Theorem)

The images of G and H inside $\mathbb{B}(\mathcal{H})$ generate each others' commutants.

Recall that left and right regular representation generate each others' commutants and we have

$$L^2(G) \simeq \int_{\widehat{G}} \pi \otimes \pi^* d\mu(\pi)$$

as a representation of $G \times G$.

In a similar spirit, we will consider the decomposition

$$\omega$$

as a representation of $G \times H$.

Local Theta Correspondence

Example ($G = \mathrm{SL}_2(\mathbb{R})$, $H = O(2p, 0)$)

We have

$$L^2(\mathbb{R}^{2p}) \simeq \sum_{n \geq 0} \mathcal{H}_n \otimes D_{n+p}$$

where $\mathcal{H}_n$ are the *spherical harmonics* representations of H

$$\mathcal{H}_n := \{\text{harmonic homog. polynomials of degree } n \text{ in } 2p\text{-variables}\}$$

and

$$D_k$$

are the holomorphic discrete series reps of G with lowest weight k.

Local Theta Correspondence

Let

$$\mathrm{Irr}(G),\ \mathrm{Irr}(H)$$

denote the *admissible* irreducible representations of G and H.

Theorem (Howe Duality, aka Local Theta Correspondence)

The rule

$$\pi \leftrightarrow \sigma \iff \mathrm{Hom}_{G\times H}(\omega^{\infty}, \pi \otimes \sigma) \neq 0$$

sets up a bijection between the sets

$$\mathrm{Irr}_{\omega}(G) := \{\pi \in \mathrm{Irr}(G) \mid \mathrm{Hom}_{G}(\omega^{\infty}, \pi) \neq 0\}$$

$$\mathrm{Irr}_{\omega}(H) := \{\sigma \in \mathrm{Irr}(H) \mid \mathrm{Hom}_{H}(\omega^{\infty}, \sigma) \neq 0\}.$$

Proven by Howe, Waldspurger, Mínguez, Gan-Takeda, Gan-Sun.

Local Theta Correspondence

We will consider special types of duals pairs (G, H) for which the theta correspondence enjoys nice features:

- **STABLE RANGE CASE:** One of the groups is

  "*at least twice as big as the other*"

  e.g. $(\mathrm{Sp}(V'), O(V))$ with $\dim(V') \geq 2\dim(V)$
  Here **unitarity** is preserved. In fact, $\widehat{H}$ embeds into $\widehat{G}$.

- **EQUAL RANK CASE:** G and H are

  "*equal in size*".

  e.g. $(\mathrm{Sp}(V'), O(V))$ with $\dim(V) = \dim(V') + 1$
  Here **temperedness** is preserved.

Results: equal rank case

Theorem (Mesland-Ş.) (equal rank)

Let (G, H) be an **equal rank** *dual pair.*

1. *The* **tempered** *local theta correspondence arises from a C^*-correspondence of the* **reduced** *group C^*-algebras of G and H.*

Results: equal rank case

Theorem (Mesland-Ş.) (equal rank)

Let (G, H) be an **equal rank** *dual pair.*

1. *The* **tempered** *local theta correspondence arises from a C^*-correspondence of the* **reduced** *group C^*-algebras of G and H.*
2. *When F is non-archimedean and (G, H) is ortho-symplectic or unitary, this C^*-correspondence can be shown to be an* **equivalence bimodule** *for suitable* **ideals** *of the reduced group C^*-algebras.*

Results: stable range case

Theorem (Mesland-Ş.) (stable range)

Let (G, H) be a **stable range** *dual pair.*

1. *The* **unitary** *local theta correspondence arises from a C^*-correspondence of the* **maximal** *group C^*-algebras of G and H.*

Results: stable range case

Theorem (Mesland-Ş.) (stable range)

Let (G, H) be a **stable range** *dual pair.*

1. *The* **unitary** *local theta correspondence arises from a C^*-correspondence of the* **maximal** *group C^*-algebras of G and H.*
2. *When F is non-archimedean and (G, H) is ortho-symplectic or unitary, this C^*-correspondence can be shown to be an* **equivalence bimodule** *for suitable* **quotients** *of the maximal group C^*-algebras.*

Some applications: transfer of characters

Recall that if π is a tempered irreducible representation of H, the character of π is the tempered distribution on H, that is, the continuous linear functional

$$\mathrm{ch}(\pi) : \mathcal{S}(H) \to \mathbb{C}$$

on Harish-Chandra's Schwartz algebra $\mathcal{S}(H)$ of H given by the trace

$$\mathrm{ch}(\pi)(\varphi) := \mathrm{tr}\ \pi(\varphi).$$

Some applications: transfer of characters

Consider an equal rank dual pair (G, H) with smooth oscillator representation

$$\omega : G \times H \to \mathrm{End}(\mathbb{S}).$$

Corollary

Let π be a tempered irreducible representation of H that enters the theta correspondence. Given $x, y \in \mathbb{S}$, let

$$_G\langle x, y\rangle \in \mathcal{S}(G), \qquad \langle x, y\rangle_H \in \mathcal{S}(H)$$

be the matrix coefficient functions defined earlier. We have

$$\mathrm{ch}(\theta(\pi))(_G\langle x, y\rangle) = \mathrm{ch}(\pi)(\langle y, x\rangle_H).$$

Note that this result was announced by Gan recently.

Some applications: preservation of formal degrees

Recall that the *formal degree* of a discrete series representation π of H is the positive real number $\mathrm{deg}(\pi)$ such that

$$\int_H \langle v, \pi(h)(v') \rangle \overline{\langle w, \pi(h)(w') \rangle} \mathrm{d}s = \frac{1}{\mathrm{deg}(\pi)} \langle v, w \rangle \langle v', w' \rangle$$

for all $v, v', w, w' \in V_\pi$.

If (G, H) is an equal rank pair, then it is known that the local theta correspondence takes discrete series representations to discrete series representations.

Some applications: preservation of formal degrees

Corollary

Let π be a discrete series representation of H which enters the theta correspondence. Then

$$\deg(\pi) = \deg(\theta(\pi)).$$

This is a known result due to Gan and Ichino.

The canonical trace is given by the orbital integral associated to the trivial conjugacy class. So one could also explicitly transfer traces arising for orbital integrals of other conjugacy classes.